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LTIPLE BINARY FORMS WITH THE CLOSURE PROPERTY.

By Arthur B. Coble.*

Introduction.

nteresting accounts of the many investigations which have centered ıt the Poncelet polygons associated with the double binary forms of r (2, 2) with the closure property have been given by G. Loria† and S. White.‡ The latter has called attention also to the existence of ıs of order (3, 3) with a similar property.§

t is the purpose of this paper to discuss the conditions for closure of neral double binary form of order (k, κ) and to indicate certain possible nsions to multiple forms. It will appear that the closure property for iven form implies the existence of a *complementary* form such that the duct of a given form and its complement can be expressed as a deter- ıant for which the closure property is inherent. A number of forms h the closure property are constructed—a number sufficient to indicate t the theory here developed has a reasonable content.

The treatment in general is algebraic though geometric interpretations constantly helpful. The striking application of the elliptic functions the Poncelet case seems not to be capable of immediate extension to ms of higher degree since the conditions for closure can not be expressed means of integrals of the first kind alone.

1. *Poristic Forms and their Complements.*—We begin with the double ıary form of order (k, κ)

$$F = (\alpha t)^k (a\tau)^\kappa = 0$$

d with one pair of values t_0, τ_0 which satisfies it. The value t_0 determines (1) κ values of τ, say $\tau_0, \tau_1, \cdots$, each of which determines $k-1$ values t in addition to t_0 and these values of t determine in turn new values of τ. roceeding in this way we obtain values $t_0, t_1, t_2, \cdots$; $\tau_0, \tau_1, \tau_2, \cdots$. If ter a finite number of repetitions of this procedure we obtain a set of values t and ν values τ such that each $t(\tau)$ determines in (1) κ (k) values $\tau(t)$ which are already included in the set of $\nu(n)$ values of $\tau(t)$ then we

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† "I polygoni di Poncelet," Torino (1889).

‡ "Poncelet polygons," *Science* (Feb. 4, 1916), pp. 149–158.

§ *Proceedings of the National Academy of Sciences*, Vol. 1 (Aug., 1915), p. 464; and ol. 2 (June, 1916), p. 337.

shall say that F *admits the configuration* $\Delta^{k,\kappa}_{n,\nu}$, $t_0, t_1, t_2, \cdots, t_{n-1}$; $\cdots, \tau_{\nu-1}$. Indeed in any geometric interpretation of the form F t set just mentioned would give rise to a geometric configuratio configuration would have moreover a type of regularity since each t's determines κ of the τ's and each of the τ's is determined by k of the It is clear from the method of derivation of the closed set that the con figuration is determined by any one of its *elements* t or of its *elements* τ.

The number of elements t which with elements τ, i.e., of *pairs* of elements t, τ, will satisfy the form F is $n\kappa$, or equally well νk, so that

$$n\kappa = \nu k. \tag{2}$$

If F admits one configuration $\Delta^{k,\kappa}_{n,\nu}$ it does not necessarily admit an infinite number. Classic examples of this are at hand. If however it does admit an infinite number the form F will be called *poristic* and t infinitude of configurations which it determines will be called a *porism*

When a configuration $\Delta^{k,\kappa}_{n,\nu}$ satisfies a form F there is determine *complementary configuration* $\Delta^{n-k,\nu-\kappa}_{n,\nu}$ made up of those pairs of elemen t, τ of the given configuration which do *not* satisfy F.

If the form F is poristic the ∞^1 sets of n elements t which belong to th ∞^1 configurations $\Delta^{k,\kappa}_{n,\nu}$ are determined by a *pencil* of binary n-ics, $(\gamma t)^n + \lambda(\delta t)^n = 0$. For evidently a set of n elements t is determined any element of the set. Similarly the ∞^1 sets of ν elements τ in the configurations are determined by a pencil of binary ν-ics, $(c\tau)^\nu + \mu(d\tau)^\nu =$ Since each set of n t's makes up with a unique set of ν τ's a configurati and conversely, these two pencils are projectively related, and for pro choice of $(c\tau)^\nu$, $(d\tau)^\nu$ in the second pencil we shall have $\mu = \lambda$. Eliminati λ between the two pencils we get

$$D_{n,\nu} = \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix} = 0. \tag{3}$$

The form $D_{n,\nu}$ from its nature as a determinant is poristic. It det mines ∞^1 *complete* configurations $\Delta^{n,\nu}_{n,\nu}$, i.e., sets of n t's and ν τ's su that each t determines *all* the τ's and vice versa. These configuratio arise from the vanishing of the general column, $\lambda_1(\gamma t)^n + \lambda_2(\delta t)^n =$ ($\lambda_1(c\tau)^\nu + \lambda_2(d\tau)^\nu = 0$, of the determinant. We shall call the form $D_{n,\nu}$ a *complete* poristic form and its configurations a *complete* porism. Clearly $D_{n,\nu}$ determined as above will contain the original poristic form as a factor and the complementary factor

$$F' = (\beta t)^{n-k}(b\tau)^{\nu-\kappa} \tag{4}$$

will also be poristic and will admit the configurations $\Delta^{n-k,\nu-\kappa}_{n,\nu}$ which a complementary to the configurations $\Delta^{k,\kappa}_{n,\nu}$ which satisfy F. Hence

(5) *Every poristic double binary form F determines a complementary poristic form F' such that each configuration of F coupled with a complementary configuration of F' makes up a configuration of the complete poristic form $FF' = D_{n,\nu}$.*

2. *Conditions for Closure. Corollaries.*—We shall now prove the theorem

(6) *The necessary and sufficient conditions that a given form $F_{k,\kappa}$ shall be poristic with configurations $\Delta_{n,\nu}^{k,\kappa}$ (where n, ν, k, κ are integers subject to the condition (2)) are (A) that the given form shall admit* ONE *such configuration, and (B) that there shall exist a form $F_{n-k,\nu-\kappa}$ which shall admit the complementary configuration.*

The necessity of these conditions appears in Sec. 1; we here prove them sufficient. Let the form F in (1) admit the $\Delta_{n,\nu}^{k,\kappa}$ with elements $t_0, t_1, \cdots, t_{n-1}$; $\tau_0, \tau_1, \cdots, \tau_{\nu-1}$. By hypothesis (B) the form F' in (4) exists such that in the product

$$FF' = (\alpha t)^k (a\tau)^k \cdot (\beta t)^{n-k} (b\tau)^{\nu-\kappa} = 0$$

each of the n t's determines all of the ν τ's and vice versa. Suppose that for $t = t_0, t_1, \cdots, t_{n-1}$ the product FF' takes the values $r_0(d\tau)^\nu, r_1(d\tau)^\nu, \cdots, r_{n-1}(d\tau)^\nu$ where $(d\tau)^\nu = 0$ has roots $\tau_0, \cdots, \tau_{\nu-1}$. We can determine (in any one of ∞^1 ways) a binary n-ic, $(\gamma t)^n$, which for $t = t_0, \cdots, t_{n-1}$ takes the values $r_0, \cdots, r_{n-1}$. Then the difference $FF' - (\gamma t)^n \cdot (d\tau)^\nu$ vanishes identically for $t = t_0, \cdots, t_{n-1}$ and therefore contains the factor $(\delta t)^n$, where $(\delta t)^n = 0$ has roots $t_0, \cdots, t_{n-1}$, and a complementary factor, say $-(c\tau)^\nu$. Hence

$$FF' = (\gamma t)^n \cdot (d\tau)^\nu - (\delta t)^n \cdot (c\tau)^\nu = \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix}.$$

But as has been pointed out the determinant is necessarily poristic and admits ∞^1 configurations $\Delta_{n,\nu}^{n,\nu}$. Since it factors into FF' each configuration $\Delta_{n,\nu}^{n,\nu}$ breaks up into configurations $\Delta_{n,\nu}^{k,\kappa}$ and $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ which satisfy respectively the forms F and F'. Hence each of the forms F and F' are poristic.

Even if the given form F admits one $\Delta_{n,\nu}^{k,\kappa}$ the number, $n(\nu-\kappa)$, of *apparent* conditions on an arbitrary form F' which ensure that it shall admit the complementary $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ is more than F' with $(n-k+1)(\nu-\kappa+1)$ coefficients will in general satisfy. However F' will certainly exist if $(n-k+1)(\nu-k+1) \geqq n(\nu-\kappa)+1$ or if

$$(7) \qquad n-k \geqq (k-1)(\nu-\kappa).$$

The cases hitherto considered relate to forms F for which $k=\kappa$ and therefore $n=\nu$. Then (7) becomes

$$(8) \qquad (2-k)(n-k) \geqq 0.$$

The case $k=\kappa=1$ presents no interest. But for $k=\kappa=2$ the relation

(8) is satisfied and the complementary form F' must exist. Thus we find the classic theorem of Poncelet:

(9) *If a form $F_{2,2}$ admits one configuration $\Delta^{2;\,2}_{n;\,n}$ it admits infinitely many and is poristic.*

Moreover it is known that poristic forms $F_{2,2}$ with configurations Δ actually exist for all values $n \geqq 3$. Hence the complementary poris forms also exist, i.e.,

(10) *There exist poristic forms $F_{n\,2,\,n-2}$ with configurations $\Delta^{n-2,\,n=2}_{n,\,n}$ fo all values $n \geqq 4$.*

The extreme cases where the orders of F' are as low as possible are worth noting. Thus if $n - k = 0 = \nu - \kappa$ and F' therefore reduces to a constant the relation (7) is satisfied and we can state that

(11) *If a form $F_{k,\,\kappa}$ admits one configuration $\Delta^{k;\,\kappa}_{k;\,\kappa}$ it is completely poristic.*

If $n - k = 1 = \nu - \kappa$ we find from (2) that $k = \kappa$. Then the complementary form F' is $(\beta t)(b\tau)$ and each of the $k + 1$ t's determines one of the $k + 1$ τ's in the projectivity $F' = 0$. Hence

(12) *If a form $F_{k,\,k}$ admits one configuration $\Delta^{k,\,k}_{k+1,\,k+1}$ it will be poristic if and only if furthermore the $k + 1$ elements t are projective to the $k + 1$ elements τ in the order indicated by the complementary configuration $\Delta^{1;\,1}_{k+1,\,k+1}$.*

This checks the theorem of Meyer which states that if a tetrahedron is inscribed in a cubic curve C with points t and circumscribed to a cubic curve C' with planes τ the curves will not in general admit ∞^1 such tetrahedra. Meyer's condition for the poristic case is that a second such tetrahedron shall exist. But since the incidence condition of point t and plane τ is $F = (\alpha t)^3(a\tau)^3 = 0$, the theorem (12) furnishes as the simpler additional condition for a porism that the four points of the given tetrahedron shall have the same double ratio on C as the four planes have on C'.

It may be observed that in the poristic case of theorem (12) we may always subject τ to such a linear transformation that the complementary form F' becomes $t - \tau$. Then the form F becomes symmetric and vanishes when t, τ belong to the same set of the involution I^{k+1}_1 determined by the pencil, $(\gamma t)^{k+1} + \lambda(\delta t)^{k+1} = 0$. Theorem (12) then reads

(13) *If a symmetric form $F_{k,\,k}$ admits one configuration $\Delta^{k,\,k}_{k+1,\,k+1}$ it admits ∞^1 such configurations.*

3. *Covariant Porisms.*—In the actual construction of poristic forms certain porisms covariantly connected with the given porism are very useful. They are defined as follows. Let the form $F_{k,\,\kappa}$ be poristic with configurations $\Delta^{k;\,\kappa}_{n;\,\nu}$ whence $FF' = D_{n,\,\nu}$. Let S, T, U denote respectively the correspondences from t to τ determined by F, F', $D_{n,\,\nu}$. We use Severi's (Lezioni) definitions for the sum, product, and inverse of these correspondences. Then

$$U = S + T, \qquad U^{-1} = S^{-1} + T^{-1},$$

$$UU^{-1} = (S + T)(S^{-1} + T^{-1}) = SS^{-1} + TT^{-1} + ST^{-1} + TS^{-1}.$$

For these products we may state at once that

(14) *The products SS^{-1}, TT^{-1}, UU^{-1} are symmetric correspondences between t and t' given respectively by forms $G_{\kappa(k-1),\,\kappa(k-1)}$, $G'_{(\nu-\kappa)(n-k-1),\,(\nu-\kappa)(n-k-1)}$, $I^{\nu}_{n-1,\,n-1}$ which express that in $F = 0$, $F' = 0$, $D_{n,\,\nu} = 0$ respectively t and t' determine a common τ. Here $I_{n-1,\,n-1} = 0$ expresses that t, t' belong to the same set of the involution $(\gamma t)^n + \lambda(\delta t)^n = 0$.*

(15) *The products ST^{-1} and TS^{-1} are correspondences between t, t' given by forms $H_{k(\nu-\kappa),\,\kappa(n-k)}$, $H'_{\kappa(n-k),\,k(\nu-\kappa)}$ which express respectively that t in $F = 0$ and t' in $F' = 0$, and t in $F' = 0$ and t' in $F = 0$, determine a common τ. The forms H, H' interchange with t, t' since ST^{-1} and TS^{-1} are inverses.*

Corresponding to the sum of products above we have at once the identity among the forms just defined

$$I^{\nu} = G \cdot G' \cdot H \cdot H'. \tag{16}$$

Evidently if the forms F, F' are poristic and have the closure property this must be true of the covariant forms I, G, G', H, H' as well and these will be called *covariant porisms.* It is clear moreover that all of these forms can be deduced from F by rational processes whence they are *rational covariants* of F.

On interchanging the order of the factors in the above products and interchanging also n, ν; k, κ; t, τ; t', τ' we have correspondences $S^{-1}S$, $T^{-1}T$, $U^{-1}U$, $S^{-1}T$, $T^{-1}S$ determined by forms $\overline{G}$, $\overline{G'}$, $\overline{I}$, $\overline{H}$, $\overline{H'}$ for which

$$\overline{I}^{n} = \overline{G} \cdot \overline{G'} \cdot \overline{H} \cdot \overline{H'}. \tag{17}$$

For a complete porism the correspondence T does not exist and $G = I^{\nu}$ but as a rule it will be necessary that the forms I and $\overline{I}$ factor and that various powers of these factors shall be distributed among the forms G, H in (16) and (17). This is true for example of the Poncelet porisms $\Delta^{2,\,2}_{n,\,n}$ $(n > 3)$ for which I is an $(n-1, n-1)$ form and G a $(2, 2)$ form.

An especially symmetric type of porism may occur when $G = I$ and $\overline{G} = \overline{I}$. Then $\kappa(k-1) = n-1$ and $k(\kappa-1) = \nu - 1$. On eliminating n, ν from (2) we find that $(\kappa - k)(\kappa - 1)(k-1) = 0$. Since $\kappa > 1$ and $k > 1$ this requires that $\kappa = k$ and therefore $n = \nu$ and $k(k-1) = n-1$. In this case the configurations are $\Delta^{r+1,\,r+1}_{r^2+r+1,\,r^2+r+1}$'s $(r = k-1)$. If $r = p^j$ where p is a prime a grouping of t's and τ's is immediately suggested by the finite plane geometry (mod p^j) where of the totality of $r^2 + r + 1$ points t and $r^2 + r + 1$ lines τ there are $r+1$ lines τ on a point t and $r+1$ points t on a line τ. For such configurations we have the theorem:

(18) *If a form* $F_{r+1,\,r+1}$ $(r > 2)$ *admits one configuration* $\Delta^{r+1,\,r+1}_{r^2+r+1,\,r^2+r+1}$ *so arranged that any two t's determine a common* τ *(or vice versa) it admits infinitely many.*

For then the symmetric form G admits a configuration $\Delta^{r^2+r,\,r^2+r}_{r^2+r+1,\,r^2+r+1}$ and according to (13) is poristic. Hence F also must have the closure property. This series of cases includes for $r = 2$ that of Professor White.* The question as to whether for values of $r > 2$ F can admit *one* configuration of the sort described in (18) is still open.

4. *The Group Problem of the Porism.*—In order that a form $F_{k,\,\kappa}$ may admit a configuration $\Delta^{k,\,\kappa}_{n,\,\nu}$ it is necessary that the n t's be assorted into ν sets of k each (with which we associate the values τ) in such a way that each t is included in κ of the ν sets. This arrangement into sets must be determined before the conditions of theorem (6) can be applied. The problem of finding such arrangementsbe longs to the theory of finite groups and the solution for given values of n, ν, k, κ is not necessarily unique. Indeed the complications of this tactical problem are the most serious bar to any general discussion of the porisms. Certain series of cases may be treated as a class. Such for example are the Poncelet polygons of n sides, and the checkerboard configurations of n^2 points of Sec. 10. As a rule however each case presents its own peculiarities. Precisely the same tactical problem appears in the formation of irrational (i.e., non-symmetric) invariants of a set of n points in an S_{k-1} of weight ν and degree κ.

If $F_{k,\,\kappa}$ is poristic as above and if we transform its variables by means of the equations

$$t = \frac{(\lambda t')^r}{(\mu t')^r}, \qquad \tau = \frac{(l\tau')^\rho}{(m\tau')^\rho}$$

then the transformed form will likewise have the closure property with configurations $\Delta^{rk,\,\rho\kappa}_{rn,\,\rho\nu}$. The peculiarity of poristic forms which arise by such transformation from simpler forms is that the nr t''s ($\nu\rho$ τ''s) divide into $n(\nu)$ sets of $r(\rho)$ each such that every $t'(\tau')$ in a set determines the same values of $\tau'(t')$. Such a grouping may be described as *imprimitive.* We shall however consider only such configurations for which the elements τ vary with t and vice versa except in the case of the complete porisms.

5. *A Geometric Source of All Poristic Double Forms.*—We shall always choose t so that, in the form $F_{k,\,\kappa}$, $k \leqq \kappa$. We can arrange the terms in $F_{k,\,\kappa}$ according to powers of t as in

$$(19) \qquad t^k(a_0\tau)^\kappa + t^{k-1}(a_1\tau)^\kappa + t^{k-2}(a_2\tau)^\kappa + \cdots + (a_k\tau)^\kappa.$$

Then if t is the parameter of an S_{k-1} u of the rational norm curve N^k in S_k,

* Cf. Coble, *Proceedings of the National Academy of Sciences*, Vol. 2 (1916), p. 530.

the ν τ's by τ_{ij} if τ_{ij} determines t_i, t_j in $F_{2,\kappa}=0$. Then for $F_{2,3}$ with a $\Delta_{4,6}^{2,3}$ we have $4t$'s, t_0, t_1, t_2, t_3 and $6\tau_{ij}$'s ($i, j = 0, \cdots, 3$; $i \neq j$). It is convenient to think of the t's as the sides and the τ's as the vertices of a complete quadrilateral such that in $F_{2,3}=0$ the sides are paired with vertices on them. In the complementary configuration $\Delta_{4,6}^{\prime 2,3}$ the sides are paired with vertices not on them. The number of pairs in each configuration is 12 and the complementary forms F, F' have 12 coefficients each so that the requirements of theorem (6) can be satisfied by imposing not more than two conditions on the $4t$'s and 6τ's.

To find these conditions we use the covariant correspondences of Sec. 3. Since $SS^{-1}=I$, $TT^{-1}=I$, $ST^{-1}=TS^{-1}=I^2$, the condition (16) is satisfied without imposing any conditions on $\Delta_{4,6}^{2,3}$. For the inverse correspondences we find however that $\bar{I}$ breaks up into two symmetric correspondences $\bar{J}\bar{K}$ such that $\bar{J}$ is satisfied by a pair τ_{ij}, τ_{ik} of adjacent vertices of the quadrilateral and $\bar{K}$ by a pair τ_{ij}, τ_{kl} of opposite vertices. Then $\bar{G}=\bar{G}'=\bar{J}$, $\bar{H}=\bar{H}'=\bar{J}\bar{K}^2$.

Since τ_{ij}, τ_{kl} satisfy the (1, 1) symmetric form $\bar{K}$ they are partners in an involution whose fixed points we shall take to be 0, ∞. Then in non-homogeneous coördinates the six τ's are

$$\begin{matrix}\tau_{01}, \tau_{02}, \tau_{03} \\ \tau_{23}, \tau_{31}, \tau_{12}\end{matrix} = \begin{matrix}\lambda, & \mu, & \nu \\ -\lambda, & -\mu, & -\nu.\end{matrix}$$

Hence the three τ's which correspond respectively to t_0, t_1, t_2, t_3 are determined by the cubics $(\tau-\lambda)(\tau-\mu)(\tau-\nu)$, $(\tau-\lambda)(\tau+\mu)(\tau+\nu)$, $(\tau+\lambda)(\tau-\mu)(\tau+\nu)$, $(\tau+\lambda)(\tau+\mu)(\tau-\nu)$. Since for all values of t in $F_{2,3}$ we obtain at most three linearly independent cubics, these four must be connected by a linear relation whose coefficients are $(\mu-\nu)(\nu-\lambda)(\lambda-\mu)$, $(\mu-\nu)(\nu+\lambda)(\lambda+\mu)$, $(\mu+\nu)(\nu-\lambda)(\lambda+\mu)$, $(\mu+\nu)(\nu+\lambda)(\lambda-\mu)$. Denote the cubics in τ with these respective coefficients by C_0, C_1, C_2, C_3 whence

$$C_0+C_1+C_2+C_3\equiv 0.$$

Let us now determine 4 quadratics in t, Q_0, Q_1, Q_2, Q_3 such that

$$Q_0+Q_1+Q_2+Q_3\equiv 0,$$

and for $t=t_i$, $Q_i=-3\rho$ and $Q_k=\rho$ $(k\neq i)$. Such quadratics are

$$Q_0=r_1+r_2+r_3, \qquad Q_i=-3r_i+r_j+r_k \quad (i, j, k=1, 2, 3),$$

$$r_i=(t_0-t_i)(t-t_j)(t-t_k).$$

Hence in

(22) $$C_0Q_0+C_1Q_1+C_2Q_2+C_3Q_3$$

we have an $F_{2,3}$ which admits the given $\Delta_{4,6}^{2,3}$. Moreover by changing the

sign of τ we get an $F'_{2,3}$ which admits the complementary $\Delta'^{2,3}_{4,6}$. Thus the conditions of theorem (6) are satisfied. The $F_{2,3}$ (22) and one of its configurations have the 9 constants, $t_0, \cdots, t_3, \lambda, \mu, \nu, 0, \infty$ whence F alone has 8. Therefore

(23) *The form $F_{2,3}$ in* (22), *subject to three conditions and containing* 8 *constants or two absolute constants is poristic and admits* ∞^1 *configurations* $\Delta^{2,3}_{4,6}$. *It is the only type which admits such configurations.*

The degenerate configurations are worth noting. Let as in Sec. 5 t be a line of the norm conic K and τ a point of the rational involution cubic curve C. Suppose that on the conic t_0 and t_1 coincide. Then τ_{01} is a common point of K and C. Moreover now the six τ's are $\left(\begin{smallmatrix} a & b & -b \\ -a & -b & b \end{smallmatrix}\right)$. Thus t_2, t_3 are lines of C since on them two intersections coincide. The pencil of sextics in τ is a pencil of cubics in τ^2 so that it happens four times that $\tau^2 = b^2$ counts twice. Hence the four configurations of the above type account for four of the six common points and the eight common lines of C and K.

In the pencil of sextics there will be either one other sextic with two double roots at $0, \infty$ or two sextics each with one double root at $0, \infty$ respectively. In the pencil of quartics t there are six which have a double root and four of these have been accounted for. If another has only one double root it leads as above to further common lines of C and K. But these have all been accounted for. Hence there is one further quartic with two double roots $t_0 = t_1$, $t_2 = t_3$ and the six τ's are $\left(\begin{smallmatrix} \tau_{01} & 0 & \infty \\ \tau_{23} & 0 & \infty \end{smallmatrix}\right)$. Hence the two tangents t_0, t_2 of K meet in the node $0, \infty$ of C and the points τ_{01}, τ_{23} of C are the two further common points of C and K. Thus C is merely the involution curve of an I^4_1 on a conic K which has acquired a node due to the fact that the I^4_1 contains a perfect square.

7. *The Poristic $F_{2,4}$ with Configurations $\Delta^{2,4}_{5,10}$.*—The case treated in the preceding section can be extended one step as indicated in the closing sentence. The involution curve of an I^5_1 on a conic K is a Lüroth quartic curve. If the involution contains three members each with two double roots then the quartic involution curve is rational. If τ_{ij} is a parameter on this rational curve C such that the tangents t_i, t_j of K meet in the point τ_{ij} of C then there exist ∞^1 configurations $\Delta^{2,4}_{5,10}$ consisting of 5 tangents of K whose 10 meets are on C, and the incidence condition of tangent t_i of K and point τ_{ij} of C, an $F_{2,4}$ form, will admit these configurations. The general I^5_1 with 8 constants is here subject to three conditions to account for the three pairs of double roots so that the poristic $F_{2,4}$ has five constants or two absolute constants.

If in a set of I^5_1, $t_0 = t_1$, $t_2 = t_3$ then the tangent t_4 of K is a double tangent of C while the contacts of tangents t_0, t_2 of K with K are points of C.

If in a set of I_1^5, $t_0 = t_1$ then the lines t_2, t_3, t_4 are common tangents of C and K and the point t_0 of K is a common point of C and K. The first case happens three times, the second case twice, so that the common points and lines of C and K are accounted for.

The existence of this case implies the existence of the complementary poristic form $F_{3,6}$ with configurations $\Delta_{5,10}^{3,6}$.

The I_1^3 (with Poncelet conics K, C), the I_1^4, and the I_1^5 are the only involutions with a sufficient number of double elements to allow the *complete* involution curve C to acquire a sufficient number of double or multiple points not on the conic K to become rational. Perhaps the series may be continued by allowing multiple points of C to appear on K. We shall however pass on to other configurations for which the involution curve C has a rational part.

8. *The Poristic $F_{2,3}$ with Configurations $\Delta_{6,9}^{2,3}$.*—There are but two distinct ways in which six t's, $t_1, \cdots, t_6$ can be arranged in nine pairs in such a manner that each t occurs in three pairs. For both ways the t's divide into two sets of three arranged as in a matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \end{pmatrix}.$$

In the one way each t is paired with a t in the same row or column to form a pair τ_{ij}. For such a configuration the form $F_{2,3}$ could not be poristic. In fact the symmetric (3, 3) form G of SS^{-1} which coördinates to the line t_1 of K the lines t_4, t_2, t_3 would break up into a (2,2) form for which t_1 determines t_2, t_3 and a (1, 1) form for which t_1 determines t_4. Thus the rational cubic C determined by G would factor into a conic and a line which would require that the form $F_{2,3}$ should factor.

The second grouping which, as we shall see, leads to a poristic $F_{2,3}$ coördinates a t in one row to each of the t's in the other row, i.e., t_i ($i = 1, 2, 3$) and t_j ($j = 4, 5, 6$) determine τ_{ij}. If t is a line of a conic K and τ_{ij} a point of the rational cubic C then the 9-point configuration τ_{ij} is cut out on C by two triads of tangents of K. There are on C ∞^4 such 9-point configuration, but of these only ∞^3 for which the six lines touch a conic. The form $F_{2,3}$ if poristic will admit ∞^1 of these configurations. Let us study under this hypothesis the necessary behavior of the covariant porisms. The form G symmetric of degree three in t, t' attached to SS^{-1} will for $t = t_1$, or t_2, or t_3 determine the triad $t' = t_4$, t_5, t_6; and for $t = t_4$, or t_5, or t_6 the triad $t' = t_1$, t_2, t_3. Hence as the $\Delta_{6,9}^{2,3}$'s more around we have a pencil of cubics $(\gamma t)^3 + \lambda(\delta t)^3$ such that this member of the pencil determines involutorily another member $(\gamma t)^3 - \lambda(\delta t)^3$. This can be visualized by taking sections of a space cubic curve by planes on a line where the planes π_1, π_2 cut out the

fixed members $(\gamma t)^3$, $(\delta t)^3$ of the above involution. Then points t, t' on the space cubic are apolar to the pair of planes π_1, π_2 if

$$G = (\gamma t)^3(\delta t')^3 + (\gamma t')^3(\delta t)^3 = 0.$$

Interpreting this form, which is evidently poristic, on K, the point t, t' runs over a cubic C of genus one. The form G has three absolute constants which arise from π_2 after the space cubic and π_1 have been chosen. Hence the cubic C and conic K together have eleven constants, or given C there are ∞^2 conics K each with $\infty^1 \Delta_{6,9}^{2,3}$'s. Thus the ∞^3 configurations whose six lines touch a conic accounted for and it is clear that the existence of one such configuration of six t's for a symmetric (3, 3) form implies the existence of infinitely many.

The cubic C must acquire a node in order that its points may be named by a parameter τ_{ij}. This requires that in one set of six t's, $t_1 = t_2$ and $t_4 = t_5$, or that π_1, π_2 be a pair of planes apolar to two tangent planes of the space cubic on the common line of π_1, π_2, which is one condition. Hence

(24) *There exist poristic forms $F_{2,3}$ with configurations $\Delta_{6,9}^{2,3}$ depending upon two absolute constants. The existence of one such configuration implies the porism.*

For given F and one configuration Δ the equation of the rational cubic C referred to K is a symmetric (3, 3) form G which admits one $\Delta_{3,3}^{3,3}$ and therefore according to (11) is poristic. This requires in turn that F be poristic.

In the pencil of cubics $(\gamma t)^3 + \lambda(\delta t)^3$ there are four members with a double root but two of these are paired at the double point so that we have one case

$$\begin{pmatrix} t_1 & t_1 & t_3 \\ t_4 & t_4 & t_6 \end{pmatrix}.$$

The double point is either $\tau_{14} = \tau_{25}$ or $\tau_{15} = \tau_{24}$ with further coincidences $\tau_{16} = \tau_{26}$ and $\tau_{34} = \tau_{35}$ so that the lines t_3, t_6 are common tangents of C and K. There remain two cases

$$\begin{pmatrix} t_1 & t_1 & t_3 \\ t_4 & t_5 & t_6 \end{pmatrix}.$$

Then $\tau_{14} = \tau_{24}$, $\tau_{15} = \tau_{25}$, $\tau_{16} = \tau_{26}$ and the lines t_4, t_5, t_6 account for the remaining common tangents of C and K. Moreover corresponding to the double cubics of the involution, $(\gamma t)^3$ and $(\delta t)^3$, there are two cases

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Then $\tau_{15} = \tau_{24}$, $\tau_{16} = \tau_{34}$, $\tau_{26} = \tau_{35}$ and the points t_1, t_2, t_3 of K are on C. Thus the common points of C and K are located. Moreover the pencil of

9-ics τ will have sixteen members with double roots and all of these appear above.

The configuration $\Delta_{6,\,9}^{2,\,3}$ is a special case of the *checkerboard configuration* of n^2 points which are the intersections of one set of n lines with another set of n lines. These are discussed further in the next two sections.

9. *The Poristic $F_{2,\,4}$ with Checkerboard Configurations $\Delta_{8,\,16}^{2,\,4}$.*—If a quartic Q has an inscribed checkerboard cut out by the two sets of four lines Q_1, Q_2 then Q has the form $Q_1 + \mu Q_2 = 0$. This form contains $2 \times 8 + 1$ or 17 constants whereas Q has but 14 so that a given quartic has presumably ∞^3 checkerboards. We should expect therefore a finite number for which the eight lines touch a conic K. But as a matter of fact if one set of eight lines touches K there will be ∞^1 sets which touch K. For if $(\gamma t)^4$ and $(\delta t)^4$ represent these sets of four lines on K, the equation of Q in Darboux coördinates t, t' is $(\gamma t)^4(\delta t')^4 + (\gamma t')^4(\delta t)^4 = 0$ and this form is poristic. This pair of binary quartics on K has five absolute constants whence Q is subject to one invariant condition. Thus we find the theorem analogous to the Lüroth theorem on the inscribed pentagons of a quartic curve:

(25) *If a quartic Q passes through the sixteen points of one checkerboard whose eight lines touch a conic K then Q is subject to one invariant condition and has ∞^1 inscribed checkerboards whose lines envelop K.*

The two sets of four lines which make up the checkerboard are pairs of sets of an I_1^4 on K which are involutorily related. If Q should have three double points and be rational with parameter τ then the condition that point τ of Q and line t of K be incident is a poristic $F_{2,\,4}$. If a binary quartic of the pencil has a double root while the paired quartic has not, i.e., if the pair of quartics is $\binom{a\ a\ b\ c}{\alpha\ \beta\ \gamma\ \delta}$ then $\alpha, \beta, \gamma, \delta$ are common tangents of K and Q. If both quartics of the pair have a double root as in $\binom{a\ a\ b\ c}{\alpha\ \alpha\ \beta\ \gamma}$ then the point (a, α) is a double point of Q and the lines b, c, β, γ are common tangents. If both quartics of the pair have a triple root as in $\binom{a\ a\ a\ b}{\alpha\ \alpha\ \alpha\ \beta}$ then the point (a, α) is a triple point of Q while b, β are common tangents of K and Q which are inflexional for the latter.

If then the case $\binom{a\ a\ b\ c}{\alpha\ \alpha\ \beta\ \gamma}$ occurs three times Q has three nodes and twelve tangents in common with K. All the multiple roots of the pencil are accounted for but clearly the six quartics of the pencil with double roots must have their parameters in involution in order that the fixed pair $(\gamma t)^4$, $(\delta t)^4$ may exist. Such a pencil has only two absolute constants.

If the case $\binom{a\ a\ a\ b}{\alpha\ \alpha\ \alpha\ \beta}$ occurs there will remain two cases $\binom{a\ a\ b\ c}{\alpha\ \beta\ \gamma\ \delta}$ so that Q has a triple point and $2.4 + 2.2$ tangents in common with K. Then $(\gamma t)^4$, $(\delta t)^4$ is any one of the ∞^1 pairs of quartics whose parameters are harmonic to the two with triple points so that for this case also there are two absolute constants.

(26) *There are two types, each with two absolute constants, of poristic $F_{2,4}$'s with configurations $\Delta^{2,4}_{8,16}$ of the checkerboard type. The checkerboards are inscribed in a quartic which for the one type has three nodes; for the other, a triple point.*

10. *The Poristic $F_{2,n}$ with Checkerboard Configurations $\Delta^{2,n}_{2n,n^2}$.*—Let us generalize the configuration of the preceding section to $2n$ lines and n^2 points and inquire as to the existence of porisms with these configurations. As before they must be associated with two particular members of a pencil of binary n-ics if the $2n$ lines envelop a conic K. An n-ic curve and one inscribed checkerboard has $4n+1$ constants. It is $2n-5$ conditions that the $2n$ lines envelop K so that then the curve and configuration have $2n+6$ constants or $2(n-1)$ absolute constants. However it is then determined by two fixed members of a pencil of binary n-ics which have $2(n-1)-1$ absolute constants. Hence

(27) *If a curve C of order n has an inscribed checkerboard of n^2 points and $2n$ lines which envelop a conic K then there are ∞^1 such checkerboards inscribed in C and circumscribed to K. The curve C has $2n-3$ absolute constants.*

Let us now attempt to make C rational by giving it nodes. Suppose that one member of the pencil of n-ics has k double roots and that the paired members has l double roots. Then each double root of the one member with each double root of the other member determines a node of C so that this pair of n-ics contributes $k+l$ of the $2(n-1)$ double roots of the pencil and contributes kl nodes to C. If C is rational for r such cases we have

$$(28)\qquad \begin{aligned} k_1+l_1+k_2+l_2+\cdots+k_r+l_r &= 2(n-1),\\ k_1l_1+k_2l_2+\cdots+k_rl_r &= (n-1)(n-2)/2. \end{aligned}$$

In each pair of this sort the $n-2k$ simple roots of the first member will each contribute an l-fold common tangent of C and K. Since it is a consequence of (28) that

$$(n-2k_1)l_1+(n-2l_1)k_1+\cdots+(n-2k_r)l_r+(n-2l_r)k_r = 4(n-1),$$

all the common tangents of C and K are found.

It is however $k-1$ conditions on the pencil that a member shall have k double roots, $l-1$ conditions that another shall have l double roots, and one condition that these be paired. Hence the number of absolute constants which remain is

$$2n-3-\sum_{i=1}^{r}(k_i+l_i-1) = 2n-3-\{2(n-1)-r\} = r-1.$$

An evident inequality for the integers k, l is

$$(29)\qquad 0 \lesseqgtr k_i \lesseqgtr n/2, \qquad 0 \lesseqgtr l_i \lesseqgtr n/2.$$

One solution of the diophantine system (28), (29) is

$$(30)\quad n = 2j+1\begin{cases} k_1,\, l_1 = j,\, j \\ k_2,\, l_2 = j,\, j-1; \\ k_3,\, l_3 = 1,\, 0 \end{cases} \qquad n = 2j\begin{cases} k_1,\, l_1 = j,\, j-1 \\ k_2,\, l_2 = j-1,\, j-1 \\ k_3,\, l_3 = 1,\, 0. \end{cases}$$

Here $r - 1 = 2$ so that these forms have two absolute constants. But there may well be other solutions of the system.

If the diophantine system above be extended to take account of the possibility of points of multiplicity as great as $n - 1$ on C, the system becomes much more complicated but solutions of this higher type exist. For example construct the pencil of n-ics by taking an n-ic with $(n-1)$-fold root a and simple root b and another with $(n-1)$-fold root α and simple root β (one absolute constant) and choose for $(\gamma t)^n$, $(\delta t)^n$ a pair of n-ics of the pencil whose parameters are apolar to the two with $(n-1)$-fold roots (a second absolute constant). Then C has an $(n-1)$-fold point at (a, α) and the tangents b, β of K have $(n-1)$-point contact with C. Hence we have the generalized form of (26).

(31) *There will exist poristic forms $F_{2,\,n}$ with configurations $\Delta^{2,\,n}_{2n,\,n^2}$ of the checkerboard type whose $2n$ lines t envelop a conic K while the n^2 points τ describe a rational n-ic curve C whose multiple points may consist only of double points as in* (30), *or of a single $(n-1)$-fold point, or of points with intermediate multiplicities. The extreme cases actually constructed above both involve two absolute constants.*

11. *The Poristic $F_{2,\,n-1}$ with Configurations $\Delta^{2,\,n-1}_{2n,\,n(n-1)}$.*—If in the checkerboard of $2n$ lines and n^2 points a diagonal be selected this implies an involutory mutual ordering of the two sets of n lines. Thus if we name the lines $\binom{t_1 t_2 \cdots t_n}{s_1 s_2 \cdots s_n}$ and the n^2 points τ_{ij} $(i, j = 1, \cdots, n)$ we may select as diagonal points the n points τ_{ii}. If this diagonal be deleted the resulting configuration will have $2n$ lines as before but only $n^2 - n$ points τ_{ij} $(i \neq j)$. If ∞^1 such configurations were circumscribed to K there would necessarily exist on K an involution I which may be taken as $t' = -t$. Then the form $(\delta t)^n$ of the preceding section would arise from $(\gamma t)^n$ by changing the sign of t. The set of $2n$ lines would have parameters $\binom{t_1\ \ t_2 \cdots\ \ t_n}{-t_1 -t_2 \cdots -t_n}$. From the curve C there would factor the line joining the fixed points 0, ∞ of the involution I on K. Of the various rational curves C referred to in (31) let us consider only the one with an $(n-1)$-fold point determined by $\binom{a,\ \ a, \cdots,\ \ a,\ \ b}{-a, -a, \cdots, -a, -b}$ for which $(\gamma t)^n = (t-a)^{n-1}(t-b)$, $(\delta t)^n = (t+a)^n(t+b)$. After factoring out the line of I there remains a curve C' of order $n-1$ with an $(n-2)$-fold point $(a, -a)$. In the equation of C' we have the one absolute constant implied in 0, ∞, a, b. If τ_{ij} is a parameter on the rational curve C' we find from the incidence condition of C' and K that

(32) *There exist poristic forms $F_{2,\,n-1}$, involving at least one absolute constant, with configurations $\Delta^{2,\,n-1}_{2n,\,n(n-1)}$ of the type t_i, s_j and τ_{ij} determined by t_i, s_j $(i, j = 1, \cdots, n;\ i \neq j)$.*

12. *The Poristic $F_{3,\,4}$ with Box Configurations $\Delta^{3,\,4}_{6,\,8}$.*—As an instance of poristic double forms with distinct orders both greater than two we take the $F_{3,\,4}$. Unless this is completely poristic with configurations $\Delta^{3,\,4}_{3,\,4}$, the simplest configurations it can admit are $\Delta^{3,\,4}_{6,\,8}$'s. For these a grouping immediately at hand is the six faces t and eight vertices τ of a cube or box where $F_{3,\,4} = 0$ coördinates to each vertex the three faces on it and to each face the four vertices on it. If we call the three pairs of opposite faces t_1, t_2; t_3, t_4; t_5, t_6, the vertices may be named τ_{135}, τ_{246}; τ_{136}, τ_{245}; τ_{145}, τ_{236}; τ_{146}, τ_{235} where τ_{ijk} is the vertex on faces t_i, t_j, t_k.

If $F_{3,\,4}$ is poristic the correspondence SS^{-1} associates $t_3^2, t_4^2, t_5^2, t_6^2$ to t_1. Hence the symmetric form G is a perfect square and $\sqrt{G}$ is a rational symmetric (4, 4) correspondence. In UU^{-1} t_1 corresponds to t_2, t_3, t_4, t_5, t_6 whence the form I factors into $J\sqrt{G}$ where J is a symmetric (1, 1) correspondence or involution whose pairs are opposite faces. Let this involution be $t' = -t$ so that the six faces may be taken as $\pm l, \pm m, \pm n$.

The correspondence $S^{-1}S$ makes τ_{135} correspond to $\tau_{136}^2\tau_{145}^2\tau_{235}^2\tau_{146}\tau_{236}\tau_{245}$ whence the form G factors into K^2L where K, L are symmetric (3, 3) forms which coördinate vertices joined by an edge or by a face diagonal respectively. Moreover $U^{-1}U$ makes τ_{135} correspond to τ_{246}; $\tau_{136}, \tau_{145}, \tau_{235}$; $\tau_{146}, \tau_{236}, \tau_{245}$ or $\bar{I} = MKL$ where $M = 0$ is the involutory correspondence between opposite vertices. Let us take M to be $\tau' = -\tau$ whence the eight τ's may be taken as $\pm\alpha, \pm\beta, \pm\gamma, \pm\delta$ and the form F must be unaltered by the simultaneous change of sign of t and τ.

In the configuration thus simplified the form F must coördinate the faces and vertices as follows:

$$l\colon -\alpha,\ \beta,\ -\gamma,\ \delta;\quad m\colon \alpha,\ \beta,\ -\gamma,\ -\delta;\quad n\colon -\alpha,\ \beta,\ \gamma,\ -\delta$$
$$-l\colon \alpha,\ -\beta,\ \gamma,\ -\delta;\quad -m\colon -\alpha,\ -\beta,\ \gamma,\ \delta;\quad -n\colon \alpha,\ -\beta,\ -\gamma,\ \delta.$$

In order to be unaltered by changing the signs of t, τ and to have the solutions indicated for $t = \pm l, \pm m$, $F_{3,\,4}$ must be

$$\begin{aligned}(33)\quad & r\{(\tau+\alpha)(\tau-\beta)(\tau+\gamma)(\tau-\delta)/(t-l)\\ &\qquad\qquad -(\tau-\alpha)(\tau+\beta)(\tau-\gamma)(\tau+d)/(t+l)\}\\ &+s\{(\tau-\alpha)(\tau-\beta)(\tau+\gamma)(\tau+\delta)/(t-m)\\ &\qquad\qquad -(\tau+\alpha)(\tau+\beta)(\tau-\gamma)(\tau-\delta)/(t+m)\} = 0.\end{aligned}$$

In addition for $t = n$ $F_{3,\,4}$ must vanish for $\tau = -\alpha, \beta, \gamma, -\delta$ and then

will have the proper solutions for $t = -n$. These conditions lead to the equations

$$\frac{r}{s} = \frac{(n+l)}{(n-m)} \frac{(\beta+\alpha)}{(\beta-\alpha)} \frac{(\gamma-\alpha)}{(\gamma+\alpha)} = -\frac{(n+l)}{(n+m)} \frac{(\beta+\alpha)}{(\beta-\alpha)} \frac{(\beta-\delta)}{(\beta+\delta)}$$

$$= -\frac{(n-l)}{(n-m)} \frac{(\gamma-\alpha)}{(\gamma+\alpha)} \frac{(\gamma+\delta)}{(\gamma-\delta)} = \frac{(n-l)}{(n+m)} \frac{(\beta-\delta)}{(\beta+\delta)} \frac{(\gamma+\delta)}{(\gamma-\delta)},$$

which are satisfied by

$$l : m : n = \frac{1}{(\gamma\alpha - \beta\delta)} : \frac{1}{(\gamma\delta - \alpha\beta)} : \frac{1}{(\alpha\delta - \beta\gamma)};$$

(34)

$$\frac{r}{s} = \frac{(\alpha+\beta)}{(\alpha+\gamma)} \frac{(\gamma+\delta)}{(\beta+\delta)} \frac{(\alpha\beta-\gamma\delta)}{(\alpha\gamma-\beta\delta)}.$$

Hence the form $F_{3,4}$ determined in (33), (34) admits one $\Delta_{6,8}^{3,4}$. Moreover the form $F'_{3,4}$ which arises from $F_{3,4}$ by the change of sign of t alone admits the complementary configuration whence according to (6) the form F is poristic. Two constants for t, τ each have been absorbed in the choice of the involutions $t' = -t$, and $\tau' = -\tau$. By proper choice of the unit points two of the four constants $\alpha, \beta, \gamma, \delta$ can be absorbed leaving two absolute constants for the form $F_{3,4}$ and one configuration, or one absolute constant for the form.

(35) *There exists a poristic form $F_{3,4}$ involving one absolute constant with configurations $\Delta_{6,8}^{3,4}$ of the box type; or there is in space a system of ∞^1 boxes whose faces envelop a space cubic curve and whose corners run over a rational quartic curve.*

With regard to this space quartic curve we remark that its sections and therefore their common apolar quartic all admit the involution $\tau' = -\tau$ and are therefore harmonic. The quartic therefore has a node and no absolute constant. Hence the constant in $F_{3,4}$ must be due to the choice of the series of boxes inscribed in the quartic and each of the ∞^1 series determines its own cubic curve. This situation perhaps deserves further study.

It is apparent that the box is the simplest instance in space of a "cellular configuration" (with one cell), the extension to space of the checkerboard configuration in the plane. It would be interesting to know whether forms F_{3,n^2} with cellular configurations Δ_{3n,n^3}^{3,n^2} containing $(n-1)^3$ cells occur. We shall not however attempt to multiply further examples of poristic forms but will close with a summary of the poristic cases which have been established thus far.

13. *Summary of Poristic Double Forms.*

	Configuration	*Group type*	*Reference*
1°	$\Delta^{2,2}_{n,n}$	Poncelet polygon	(9)
2°	$\Delta^{n-2,n-2}_{n,n}$	Complement of 1°	(10)
3°	$\Delta^{k,\kappa}_{k,\kappa}$	Complete porism	(11)
4°	$\Delta^{k,k}_{k+1,k+1}$	Transform of I_1^{k+1}	(12,) (13)
5°	$\Delta^{3,3}_{7,7}$	White's porism	(18)
6°	$\Delta^{4,4}_{7,7}$	Complement of 5°	
7°	$\Delta^{2,3}_{4,6}$	Quadrilateral porism	(23)
8°	$\Delta^{2,4}_{5,10}$	Pentagonal porism	Sec. 7
9°	$\Delta^{3,6}_{5,10}$	Complement of 8°	
10°	$\Delta^{2,n}_{2n,n^2}$	Checkerboard	(24), (26), (31)
11°	$\Delta^{2,n-1}_{2n,n(n-1)}$	Checkerboard without diagonal	(32)
12°	$\Delta^{2(n-1),n(n-1)}_{2n,n^2}$	Complement of 10°	
13°	$\Delta^{2(n-1),(n-1)^2}_{2n,n(n-1)}$	Complement of 11°	
14°	$\Delta^{3,4}_{6,8}$	Box porism	(35)

14. *Extensions to Multiple Forms.*—The double binary forms may be extended in various ways. One natural extension of the (2, 2) incidence condition of line t and point τ of two conics is to the incidence condition of plane and point of two quadrics. We should then inquire as to the existence of polyhedra circumscribed to the one quadric and inscribed in the other. However on naming the planes of one quadric by the parameters (of its generators) λ, μ, and the points of the other by λ', μ' the incidence condition appears in the form

$$(a\lambda)(b\mu)(c\lambda')(d\mu') = 0$$

in which unfortunately the four variables are not on a par.

More interesting, at least from the algebraic point of view, are the triple binary forms,

$$(a\lambda)^l(b\mu)^m(c\nu)^n = 0. \tag{36}$$

Let λ_0, μ_0, ν_0 be any solution of (36). Then given λ_0, μ_0 we obtain $n-1$ values of ν in addition to ν_0 which satisfy (36); and from any one of these n solutions new solutions with different λ or with different μ are obtained. If after a time no *new* solutions can be obtained by this process from those already known then we shall have a triple form with the *closure* property.

Two types of triple forms with the closure property are suggested by earlier cases. The first is an extension of the form G of Sec. 8. Let the parameters λ, λ', λ'' which determine n-ics in the pencils

$$(\gamma t)^n + \lambda(\delta t)^n = 0, \qquad (\gamma t')^n + \lambda'(\delta t')^n = 0, \qquad (\gamma t'')^n + \lambda''(\delta t'')^n = 0$$

be connected by the involutory relation

$$a_0\lambda\lambda'\lambda'' + a_1(\lambda'\lambda'' + \lambda''\lambda + \lambda\lambda') + a_2(\lambda + \lambda' + \lambda'') + a_3 = 0.$$

On replacing the λ, λ', λ'' by their values in terms of t, t', t'' we obtain a triple (n, n, n) form with the closure property.

Another triple form with the closure property is the determinant

$$\Delta = \begin{vmatrix} (a_0\lambda)^l & (a_1\lambda)^l & (a_2\lambda)^l \\ (b_0\mu)^m & (b_1\mu)^m & (b_2\mu)^m \\ (c_0\nu)^n & (c_1\nu)^n & (c_2\nu)^n \end{vmatrix} = 0 \tag{37}$$

obtained by eliminating the line coördinates ξ_0, ξ_1, ξ_2 in S_2 from the equations

$$\begin{aligned} (a_0\lambda)^l\,\xi_0 + (a_1\lambda)^l\,\xi_1 + (a_2\lambda)^l\,\xi_2 &= 0, \\ (b_0\mu)^m\xi_0 + (b_1\mu)^m\xi_1 + (b_2\mu)^m\xi_2 &= 0, \\ (c_0\nu)^n\,\xi_0 + (c_1\nu)^n\,\xi_1 + (c_2\nu)^n\,\xi_2 &= 0, \end{aligned} \tag{38}$$

which represent rational curves in S_2 of orders l, m, n and parameters λ, μ, ν respectively. The condition $\Delta = 0$ expresses that points λ, μ, ν respectively on these curves are cóllinear and obviously Δ has the closure property.

If Δ factors then each factor will likewise have the closure property. Special cases of such factorization are furnished by the involution determined by the rational plane curve of order k, $x_i = (d_i t)^k$ $(i = 0, 1, 2)$. The condition that points t_0, t_1, t_2 of this curve lie on a line is the vanishing of the determinant

$$|(d_i t_j)^k| = (t_0t_1)(t_0t_2)(t_1t_2)\cdot(\alpha_0t_0)^{k-2}(\alpha_1t_1)^{k-2}(\alpha_2t_2)^{k-2}.$$

Here the symmetric $(k-2, k-2, k-2)$ form has the closure property. Similarly if in Δ, $(b_i\mu)^m_{\mu=\nu} = (c_i\nu)^n$, then the factor $(\mu\nu)$ appears and the remaining factor is an $(l, m-1, m-1)$ form symmetric in μ, ν with the closure property. It is clear however from a discussion of these special cases* that the existence of one closed set is not in general sufficient to ensure closure.

If in the form (36) λ, μ, ν be regarded as parameters in three plane pencils on lines L, M, N respectively in S_3, and λ, μ, ν as the coördinates of the intersection of the three planes, then (36) is the equation of a surface of order $l + m + n$ with the lines L, M, N as l-, m-, n-fold lines respectively. If the form has the closure property then the points of the surface divide into closed sets such that the bisecants from any point of the set across any two of the lines L, M, N meet the surface again in points of the set. A striking case of this sort is that of the (2, 2, 2) form for which the bisecant construction for lines M, N leads to an involutory birational transformation I_L of the sextic surface into itself. The closure case occurs when the three involutions I_L, I_M, I_N generate a finite group.

* Cf. Coble, "Symmetric Binary Forms and Involutions," this JOURNAL, Vol. 32 (1910), p. 333; in particular (104) p. 336.

Three instances of such sextic surfaces may be given. If first in (37) $l = m = n = 2$ the involutions I_L, I_M, I_N generate an abelian G_8. The closed sets of eight points are cut out by three pairs of planes on L, M, N. If second in (37) $l = 2$, $m = n = 3$, and $(b_i\mu)^3_{\mu=\nu} = (c_i\nu)^3$ then the quadric $(\mu\nu)$ factors out leaving a sextic surface with closed sets of 12 points. Now I_μ and I_ν generate a dihedral G_6 and I_L, interchangeable with I_M and I_N generates with G_6 a dihedral G_{12}. If thirdly in (37) $l = m = n = 4$, $(c_i\nu)^4_{\nu=\lambda} = (a_i\lambda)^4$ and, $(b_i\mu)_{\mu=\lambda} = (a_i\lambda)^4$ then the sextic $(\mu\nu)(\nu\lambda)(\lambda\mu)$ factors out leaving a sextic surface with closed sets of 24 points. The involutions now generate a G_{24} isomorphic with the symmetric G_{24}.

These fairly evident cases where factorization of Δ is possible can be generalized to higher orders or to larger numbers of variables. Though I have not been able to find other types of factorization it would seem to be quite likely that such new types exist and their determination presents a fascinating problem.

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EINSTEIN'S THEORY OF GRAVITATION: DETERMINATION OF THE FIELD BY LIGHT SIGNALS.

BY EDWARD KASNER.

In this first paper we discuss the determination of a four-dimensional manifold

$$(1)\qquad ds^2 = \Sigma g_{ik} dx_i dx_k. \qquad (i, k = 1, 2, 3, 4)$$

obeying Einstein's equations of gravitation $G_{\mu\nu} = 0$, when we are given merely *the light equation*

$$\Sigma g_{ik} dx_i dx_k = 0.$$

This means that the ratios of the ten potentials g_{ik} are given and the problem is to determine the potentials themselves so that the gravitational equations are fulfilled.

The result shows that for example the solar gravitational field, or any field assumed to differ only slightly from the galilean field or flat space, can be completely explored by light signals alone. The corresponding determination by orbits alone will be given later. *The light rays determine the orbits, and vice versa.* A statement of results for both problems was given in *Science,* October 29, 1920 (vol. 52, pp. 413–14), in particular the connection between the planetary and light observations for the solar field.

We have also shown that the (exact) solar field can be regarded as immersed in a flat space of 6 dimensions; but that no solution of the Einstein equations can be obtained from flat space of 5 dimensions. (Cf. abstracts in *Bull. Amer. Math. Soc.*, vol. 27, pp. 62 and 102.)

§ 1. GENERAL EQUATIONS.

In order to write the general equations of gravitation we need the following symbolism.

Let g denote the determinant of fourth order formed from the coefficients $g_{\mu\nu}$; and let $g^{\mu\nu}$ denote the minor of the element $g_{\nu\mu}$ divided by g.

The Christoffel three-index symbol of the first kind is

$$(2)\qquad [\alpha\beta, \gamma] = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}\right);$$

and that of the second kind is

$$(3)\qquad \{\alpha, \beta, \gamma\} = g^{\gamma\epsilon}[\alpha\beta, \epsilon],$$

where, by the regular convention of the Einstein literature, summation is to be understood, the repeated index ϵ stating the values 1, 2, 3, 4.

The Riemann-Christoffel curvature tensor is

$$(4)\quad B^{\rho}_{\mu\nu\sigma} = \{\mu\sigma, \epsilon\}\{\epsilon\nu, \rho\} - \{\mu\nu, \epsilon\}\{\epsilon\sigma, \rho\} + \frac{\partial}{\partial x_\nu}\{\mu, \sigma\rho\} - \frac{\partial}{\partial x_\sigma}\{\mu\nu, \rho\},$$

which is also denoted by the four-index symbol $\{\mu\rho, \sigma\nu\}$.

The "contracted" Riemann-Christoffel tensor (which might well be termed the Einstein tensor) is then

$$(5)\qquad G_{\mu\nu} = B^{\rho}_{\mu\nu\rho},$$

summation understood with respect to ρ.

This can be reduced to

$$(6)\qquad G_{\mu\nu} = \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} - \frac{\partial}{\partial x_\alpha}\{\mu\nu, \alpha\} + \frac{\partial^2 L}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \alpha\}\frac{\partial L}{\partial x_\alpha},$$

where summations are taken with respect to α and β, and where

$$L = \log \sqrt{-g}.$$

In the actual world with real coördinates, the determinant g is negative since the fundamental quadratic form has three negative dimensions and one positive dimension (as in the Lorentz-Minkowski world based on the affine-euclidean form

$$dt^2 - dx^2 - dy^2 - dz^2,$$

that is, the special relativity theory). The results which follow are not dependent on this reality assumption. We may write in general

$$L = \tfrac{1}{2}\log g,$$

since only the derivatives of L are involved.

Einstein's law of gravitation (in empty space, the only case we shall here consider) is expressed by the set of ten equations

$$G_{\mu\nu} = 0,$$

involving the second derivatives of the ten functions g_{ik}. [This set takes the place of the single equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0,$$

which is the basis of the Newtonian theory.]

Any manifold (1) obeying these equations will be said to be of the Einstein type. "Euclidean space" is a special case characterized by the

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vanishing of the curvature tensor (4). More exactly we should here employ the term flat space since we must include the affine-euclidean as well as the ordinary euclidean species. Flat space thus means that the potentials g_{ik} in (1) can be reduced to constants, or that there is no permanent gravitation.

§ 2. EUCLIDEAN OR FLAT SPACE.

We now show that the light equation of an Einstein field cannot reduce to the simple form $dt^2 - dx^2 - dy^2 - dz^2 = 0$ *unless there is no permanent gravitation.*

For this purpose we use the more symmetric form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = 0,$$

and prove that the curvature tensor necessarily vanishes.

We proceed to calculate the gravitational equations $G_{\mu\nu} = 0$ in the case where the manifold is of the form

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \tag{7}$$

where λ is an unknown function of the four coördinates. The potentials are here

$$g_{11} = g_{22} = g_{33} = g_{44} = \lambda,$$

the remaining six, g_{12}, g_{13}, etc., all vanishing. The determinant g reduces to λ^4; therefore

$$g^{11} = g^{22} = g^{33} = g^{44} = \lambda^{-1},$$

the other six, g^{12}, etc., also vanishing.

We thus find immediately the first symbols

$$[11, 1] = \tfrac{1}{2}\lambda_1, \qquad [11, 2] = -\tfrac{1}{2}\lambda_2,$$
$$[12, 1] = \tfrac{1}{2}\lambda_2, \qquad [12, 3] = 0,$$

where subscripts denote partial derivatives; and the second symbols

$$\begin{aligned} \{11, 1\} &= N_1, \qquad \{11, 2\} = -N_2, \\ \{12, 1\} &= N_2, \qquad \{12, 3\} = 0 \end{aligned} \tag{8}$$

where, for convenience, we introduce

$$N = \tfrac{1}{2}\log\lambda. \tag{8'}$$

The function L occurring in the Einstein equations is then

$$L = \log\lambda^2 = 2\log\lambda = 4N.$$

Since our problem is symmetric in the four coördinates it will be sufficient to calculate G_{12} (case of unlike subscripts) and G_{11}, (case of like subscripts).

$$G_{12} = \{1\alpha, \beta\}\{2\beta, \alpha\} - \{12, \alpha\}_\alpha + L_{12} - \{12, \alpha\}L_\alpha.$$

The first term on the right is a summation of 16 terms, of which 10 vanish and 6 reduce each to N_1N_2; the second term is a sum of 4 terms contributing $-2N_{12}$; the third term is directly $4N_{12}$; the last term is a sum of 4, contributing $-8N_1N_2$. We have then

$$\begin{aligned} G_{12} &= 6N_1N_2 - 2N_{12} + 4N_{12} - 8N_1N_2 \\ &= 2N_{12} - 2N_1N_2. \end{aligned}$$

The like subscript case is a little more complicated:

$$\begin{aligned} G_{11} &= \{1\alpha, \beta\}\{1\beta, \alpha\} - \{11, \alpha\}_\alpha + L_{11} - \{11, \alpha\}L_\alpha \\ &= 4N_1^2 - N_2^2 - N_3^2 - N_4^2 - N_{11} + N_{22} + N_{33} + N_{44} \\ &\quad + 4N_{11} - 4N_1^2 + 4N_2^2 + 4N_3^2 + 4N_4^2 \\ &= 3N_{11} + N_{22} + N_{33} + N_{44} + 2(N_2^2 + N_3^2 + N_4^2). \end{aligned}$$

Hence the conditions (necessary and sufficient) that a manifold of the form

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$$

shall be an Einstein manifold (that is, $G_{\mu\nu} = 0$*) are*

$$(9)\quad \begin{aligned} &N_{12} - N_1N_2 = 0, \qquad \text{etc.}, \\ &3N_{11} + N_{22} + N_{33} + N_{44} + 2(N_2^2 + N_3^2 + N_4^2) = 0, \qquad \text{etc.} \end{aligned}$$

This is a set of 10 partial differential equations of the second order in one unknown function $N = \frac{1}{2}\log\lambda$.

It remains to show that the manifolds thus obtained are euclidean. This can be done either by actually integrating the above set, or by proving that the curvature tensor vanishes.

The easiest way to integrate the set is to use the transformation

$$(10')\qquad N = -\log M.$$

This gives

$$N_1 = -\frac{M_1}{M}, \qquad N_{12} = \frac{M_1M_2 - MM_{12}}{M^2}, \qquad N_{11} = \frac{M_1^2 - MM_{11}}{M^2}.$$

Hence the transformed set is

$$(10)\quad \begin{aligned} &M_{12} = 0, \qquad M_{13} = 0, \qquad \text{etc.} \\ &M_{11} = M_{22} = M_{33} = M_{44}. \end{aligned}$$

We easily find the general solution to be

$$(11)\quad M = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + a_1x_1 + a_1x_2 + a_3x_3 + a_4x_4 + a_5$$

with the condition

$$(11') \qquad a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4aa_5 = 0,$$

so that the result involves five arbitrary constants.

Since $N = \frac{1}{2}\log\lambda$, it follows that $\lambda = M^{-2}$; hence our manifold can be written

$$(12) \qquad ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2}{M^2}.$$

This can be reduced by linear transformation of the coördinates to one of the three forms

$$(13) \qquad \frac{\Sigma dx_i^2}{(\Sigma x_i^2)^2}, \qquad \frac{\Sigma dx_i^2}{(x_1 + ix_2)^2}, \qquad \Sigma dx_i^2,$$

which are found to be flat manifolds.

In second method we avoid the integration of set (9); and calculate instead the uncontracted curvature tensor $\{\alpha\beta, \gamma\delta\}$. Of these symbols it is sufficient to consider the types

$$(14) \qquad \begin{array}{lll} \{11, 12\}, & \{12, 21\}, & \{11, 23\}, \\ \{12, 31\}, & \{21, 31\}, & \{12, 34\}. \end{array}$$

The first, third, and sixth vanish identically. The others give conditions of the types

$$(15) \qquad \begin{array}{ll} N_{12} - N_1N_2 = 0, & \text{etc.} \\ N_{11} + N_{22} + N_3^2 + N_4^2 = 0, & \text{etc.} \end{array}$$

These are the conditions (necessary and sufficient) that

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \qquad N = \tfrac{1}{2}\log\lambda,$$

be euclidean.

It is obvious a priori that when this set is fulfilled so is the former set (9), since of course euclidean space obeys Einstein's equations. It is not obvious that the converse also holds; but it is a fact since we can verify directly that each set of ten equations in N is linearly reducible to the other set.

Hence the only manifolds of the form (7) *which obey Einstein's gravitational equations are (affine) euclidean.*

We may state this in more geometric form by noting that the only four-dimensional spreads which can be conformally represented on four-dimensional flat space are precisely those of the above form.

Hence of all the four-dimensional spaces which are conformally representable on flat space, the only ones which are of the Einstein type are (affine) euclidean.

This means that when a conformal representation of an Einstein manifold on a flat space is possible, the manifold is isometric to flat space.

§ 3. NEARLY-EUCLIDEAN MANIFOLDS.

We proceed to generalize our result to curved Einstein spaces; namely, to show that the light equation determines the space, or that two conformally equivalent Einstein spaces are applicable. We shall here give the proof for the case where the Einstein spaces differ infinitesimally from flat space.

By a *nearly-euclidean* (or approximately flat) space we understand

$$ds^2 = \Sigma dx_i^2 + \Sigma h_{ik} dx_i dx_k, \tag{16}$$

where the h's denote 10 functions (of the coördinates x_1, x_2, x_3, x_4), which functions together with their derivatives are supposed to be small, so that their squares and products are neglected. We could write instead

$$\Sigma dx_i^2 + \epsilon \Sigma h_{ik} dx_i dx_k,$$

where ϵ is a small number and the h's are arbitrary finite functions; but the first notation is more convenient.

The curvature tensor and the Einstein tensor then reduce to *linear* differential expressions of the second order. Of the 10 Einstein equations it is sufficient to write those corresponding to G_{12} and G_{11}. These are (the last two subscripts denoting partial differentiation)

$$\begin{aligned} &h_{13,\,23} + h_{23,\,13} - h_{12,\,33} + h_{14,\,24} + h_{24,\,14} - h_{12,\,44} \\ &\qquad\qquad\qquad\qquad\qquad\qquad - h_{33,\,12} - h_{44,\,12} = 0, \\ &2(h_{12,\,12} + h_{13,\,13} + h_{14,\,14}) - h_{11,\,22} - h_{11,\,33} - h_{11,\,44} - h_{22,\,11} \\ &\qquad\qquad\qquad\qquad\qquad\qquad - h_{33,\,11} - h_{44,\,11} = 0. \end{aligned} \tag{17}$$

In deriving these it is sufficient to note that for the above ds^2 we have

$$\begin{aligned} g_{11} &= 1 + h_{11}, & g_{12} &= h_{12}, \\ g &= 1 + \Sigma h_{ii}, & L &= \tfrac{1}{2}\Sigma h_{ii}, \\ g^{11} &= 1 - h_{11}, & g^{12} &= -h_{12}, \end{aligned} \tag{17$'$}$$

with the three-index symbols (both kinds here have same values)

$$\begin{aligned} \{11, 1\} &= \tfrac{1}{2} h_{11,\,1}, & \{11, 2\} &= h_{12,\,1} - \tfrac{1}{2} h_{11,\,2}, \\ \{12, 1\} &= \tfrac{1}{2} h_{11,\,2}, & \{12, 3\} &= \tfrac{1}{2}(h_{13,\,2} + h_{23,\,1} - h_{12,\,3}). \end{aligned} \tag{17$''$}$$

§ 4. The Light Equation of a Nearly-Euclidean ds^2.

This is found by putting ds^2 equal to zero; that is

$$\Sigma dx_i^2 + \Sigma h_{ik} dx_i dx_k = 0. \tag{18}$$

If a second manifold determined by 10 new functions H_{ik} is to have the same light equation, that is, if

$$\Sigma dx_i^2 + \Sigma H_{ik} dx_i dx_k = 0 \tag{19}$$

is to differ from the former equation merely by a factor λ, this factor must obviously differ only slightly from unity, that is,

$$\lambda = 1 + \mu,$$

where μ is a small function (of the same order as the h's). This shows that

$$\begin{aligned} H_{ii} &= h_{ii} + \mu \qquad (i = 1, 2, 3, 4), \\ H_{ij} &= h_{ij} \qquad (i \neq j). \end{aligned} \tag{20}$$

Suppose now that both spaces are of the Einstein type. Our problem is to find the conditions on the function μ which follow from the fact that the functions h and also the functions H obey the linear equations of second order (17). Substituting and subtracting we find these 10 equations

$$\begin{aligned} \mu_{12} &= 0, \qquad \text{etc.}, \\ 3\mu_{11} + \mu_{22} + \mu_{33} + \mu_{44} &= 0, \qquad \text{etc.} \end{aligned} \tag{21}$$

The last four equations show that

$$\mu_{11} = \mu_{22} = \mu_{33} = \mu_{44} = 0; \tag{21'}$$

hence all the second derivatives of μ vanish. It follows that the function is linear, say

$$\mu = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + b. \tag{22}$$

Our result is that if ds^2 defines any given nearly-euclidean Einstein space, then the only other possible nearly-euclidean Einstein spaces which have the same light equation are defined by

$$dS^2 = (1 + \mu) ds^2, \tag{23}$$

where μ is a linear function (22).

It remains to show that these spaces are equivalent (isometric) to the original (18). To do this, consider the general infinitesimal point transformation

$$X_i = x_i + \xi_i, \tag{24}$$

where the ξ's denote small functions of the four coördinates.

We have

$$dX_i = dx_i + \xi_{i,j}dx_j \tag{24'}$$

(summation respect to j understood); therefore

$$\begin{aligned}\Sigma dX_i^2 &= \Sigma dx_i^2 + 2\xi_{i,k}dx_idx_k \\ &= \Sigma dx_i^2 + 2\xi_{1,1}dx_1^2 + \text{etc.} + 2(\xi_{1,2} + \xi_{2,1})dx_1dx_2 + \text{etc.}\end{aligned} \tag{24''}$$

We first inquire when this reduces to $(1 + \mu)\Sigma dx_i^2$. The conditions are

$$\begin{gathered}\xi_{1,1} = \xi_{2,2} = \xi_{3,3} = \xi_{4,4} = \mu, \\ \xi_{1,2} + \xi_{2,1} = 0, \quad \text{etc.}\end{gathered} \tag{25}$$

We find that with the above value (22) of μ this set of 10 equations for the 4 unknown functions is consistent, the complete solution being

$$\begin{aligned}\xi_1 &= x_1\Sigma a_ix_i - \tfrac{1}{2}a_1\Sigma x_i^2 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 + bx_1 + B_1, \\ \xi_2 &= x_2\Sigma a_ix_i - \tfrac{1}{2}a_2\Sigma x_i^2 + A_{21}x_1 + A_{23}x_3 + A_{24}x_4 + bx_2 + B_2, \\ \xi_3 &= \\ \xi_4 &= \end{aligned} \tag{26}$$

where $A_{21} + A_{12} = 0$, etc., so that the result depends on 15 arbitrary constants. This is recognized as the general infinitesimal conformal transformations in flat space of four dimensions.

Apply this transformation to (16). The result, neglecting terms containing products of h's and ξ's, is found to be

$$(1 + \mu)\Sigma dx_i^2 + \Sigma h_{ik}dx_idx_k;$$

which may be rewritten, in virtue of (20),

$$\Sigma dx_i^2 + \Sigma H_{ik}dx_idx_k.$$

This shows that the two quadratic forms, one determined by the h's, the other by the H's of (20) are equivalent when μ has the linear form (22). We may therefore state the

THEOREM. *If two nearly-euclidean spaces both obey Einstein's equations and have the same light equation, they are necessarily equivalent* (*isometric*).

A geometric restatement would be:

If two nearly-euclidean spaces of the Einstein type are capable of conformal representation, they are necessarily isometric.

§ 5. DETERMINATION BY QUADRATURES.

Suppose we are given the light equation of some unknown Einstein space; how shall we find that space. *This can always be done by differentiations*

and quadratures. For suppose that the ten functions h_{ik} in the equation (18) do not satisfy, as they are given, the linear gravitational equations (17); the problem is then to find a function μ so that the ten functions H_{ik} found by (20) shall satisfy (17). The ten equations in μ thus obtained can be solved for the second derivatives μ_{ik}, which are thus expressed as known functions of the x's. The equations, by assumption consistent, can therefore be solved by quadratures. The result is determined up to an additive linear term which does not essentially affect the quadratic form ds^2 obtained.

§ 6. GENERAL EINSTEIN MANIFOLDS.

If we do not assume our manifolds to be nearly-euclidean, the discussion is of course more difficult since we have then to face the exact non-linear expressions (6). We shall confine ourselves here to the statement that we are lead to a set of ten non-linear equations of the second order for the unknown factor λ; these can be solved for the ten second derivatives in terms of the first derivatives; existence theorems then show that *the solution depends on not more than five arbitrary constants* (one of these is trivial, being merely a constant factor). In the special case of § 2 it was easy to show that the ∞^5 quadratic forms actually obtained were all equivalent. The general discussion will be left for a later paper.

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NOTE ON EINSTEIN'S EQUATION OF AN ORBIT.

By F. Morley.

The differential equation of an orbit is given in Eddington's "Report on the Relativity Theory of Gravitation"* as

$$(1)\qquad \left[\frac{d}{d\varphi}\frac{1}{r}\right]^2 = 2m/r^3 - 1/r^2 + 2m/h^2r - (1 - c^2)/h^2,$$

r and φ being the polar coördinates of the planet. Forsyth† has discussed the equation in connection with elliptic functions. The matter is of so great interest that a second elliptic handling seems worth publication.

We have, from Eddington,

$$c^2 = 1 - m/a, \qquad h^2 = ma(1 - e^2),$$

where a is the "major semi-axis" and e the eccentricity; therefore

$$1 - c^2 = m^2(1 - e^2)/h^2.$$

The number m^2/h^2 is, in Eddington's units, about 10^{-8}. Call it α, so that

$$1 - c^2 = \alpha(1 - e^2).$$

In the calculations α^2 will be neglected when lower powers are present.

Writing $x = m/r$, the equation becomes

$$(2)\qquad \left(\frac{dx}{d\varphi}\right)^2 = 2x^3 - x^2 + 2\alpha x - \alpha(1 - e^2).$$

The roots of the cubic on the right are $\frac{1}{2}$, 0, 0 when $\alpha = 0$. Hence writing $x = \frac{1}{2} + k_1\alpha$, $k_2\alpha$, $k_3\alpha$ and determining the coefficients k, the roots are

$$x_1 = \tfrac{1}{2} + 2\alpha, \qquad x_2 = \alpha(1 - e), \qquad x_3 = \alpha(1 + e).$$

To bring equation (2) into Weierstrass's form

$$\left(\frac{dp}{dv}\right)^2 = 4p^3 - g_2p - g_3$$

we write $d\varphi = \sqrt{2}dv$, $x = p + 1/6$. The invariants g_2 and g_3 are for a quartic with coefficients a, b, c, d, e

$$ae - 4bd + 3c^2$$

* 2d Edition (1920), p. 50. Einstein, Berlin-Sitzungsberichte, 1915, p. 831.

† Proc. Royal Society, series A, vol. 97, April, 1920.

and

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Here a, b, c, d, e are respectively 0, 1, $-\frac{1}{3}$, α, $-2\alpha^2(1-e^2)$, so that, accurately,

(3) $$g_2 = \tfrac{1}{3}(1 - 12\alpha),$$

(4) $$g_3 = \tfrac{1}{27}[1 - 18\alpha + 54\alpha^2(1-e^2)].$$

Hence the discriminant Δ, or $g_2^3 - 27g_3^2$, is

(5) $$\Delta = 4e^2\alpha^2 - 64\alpha^3.$$

The roots being real, the network or lattice which gives rise to the function p is rectangular. The essence of the problem is the determination of a period-rectangle. Let $2\omega_1$ be the real positive period, $2\lambda i\omega_1$ the imaginary period, and write as usual

$$q = e^{-\pi\lambda}.$$

The number λ is positive, and it may be called the shape of the rectangle.

The invariants g_2, g_3, are connected with the periods $2\omega_1$ and $2\omega_2$ by the equations

$$g_2 = 60\sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^4}, \qquad g_3 = 140\sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^6},$$

summed for all integer pairs m_1, m_2, the pair 0, 0 excluded. For calculation the double sums are expressed as single sums or products; and the appropriate formulæ are given in works on the elliptic functions.* But as the proofs of the full formulæ are necessarily complicated, I shall interpolate a proof of the approximate formulæ of a kind that is at once intelligible.

We have, n being any integer,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^2} = \frac{\pi^2}{\sin^2 \pi x}.$$

Differentiating,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^3} = \frac{\pi^3 \cos \pi x}{\sin^3 \pi x}.$$

Differentiating again,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^4} = \pi^4 \frac{\cos^2 \pi x + \frac{1}{3}\sin^2 \pi x}{\sin^4 \pi x} = \pi^4 \left[\frac{1}{\sin^4 \pi x} - \frac{2}{3\sin^2 \pi x}\right].$$

This enables us to sum the series for g_2, when m_2 is fixed. When $m_2 = 0$,

* They may be found in Harkness and Morley, "Theory of Functions," p. 321, formulæ 75 and 76, and p. 324, formula 82.

we get the first tier

$$\frac{2\cdot 60}{(2\omega_1)^4}\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]=\frac{15}{2\omega_1^4}\cdot\frac{\pi^4}{90}=\frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4.$$

When $m_2 = 1$, we get the second tier

$$\sum_n \frac{1}{(2\omega_2+2m_1\omega_1)^4}=\frac{1}{(2\omega_1)^4}\sum\frac{1}{(i\lambda+m_1)^4}$$

$$=\left(\frac{\pi}{2\omega_1}\right)^4\left(\frac{1}{\sin^4\pi i\lambda}-\frac{2}{3\sin^2\pi i\lambda}\right).$$

Now

$$\sin\pi i\lambda=\left(\frac{e^{-\pi\lambda}-e^{\pi\lambda}}{2i}\right)=\left(q-\frac{1}{q}\right)\Big/2i.$$

Hence the tier $m_2 = 1$ gives

$$60\left(\frac{\pi}{2\omega_1}\right)^4\left[\frac{16q^4}{(1-q^2)^4}+\frac{8q^2}{3(1-q^2)}\right]$$

or, neglecting q^4,

$$10\left(\frac{\pi}{\omega_1}\right)^4 q^2.$$

For the tier $m_2 = -1$ we get the same result. For the tiers $m_2 = \pm 2$ we have to replace q by q^2 and so on. Hence the formula

$$\left(\frac{\omega_1}{\pi}\right)^4 g_2=\tfrac{1}{12}+20q^2+\cdots. \tag{6}$$

A similar argument, with further differentiation, gives

$$\left(\frac{\omega_1}{\pi}\right)^6 g_3=\tfrac{1}{216}-\tfrac{7}{3}q^2+\cdots \tag{7}$$

and from these

$$\left(\frac{\omega_1}{\pi}\right)^{12}\Delta=q^2+\cdots. \tag{8}$$

From the formulæ (6) and (8) we have approximately

$$\Delta/g_2^3 \doteq 12^3 q^2$$

and on the other hand from (3) and (5) we have

$$\Delta/g_2^3 \doteq 4\cdot 27\cdot e^2\alpha^2.$$

Hence approximately

$$q^2 \doteq e^2\alpha^2/4^2,$$

$$q \doteq e\alpha/4.$$

For Mercury the eccentricity e is approximately $1/5$, so that

$$q \doteq 10^{-8}/20.$$

Calculating $\pi\lambda \doteq \log 1/q = 9 \log 10 + \log 2$, the shape λ of the rectangle of the periods turns out to be nearly 7. As v travels round the rectangle of half periods, p takes all real values, so that the radius vector

$$r = m/x = m/(p + \tfrac{1}{6})$$

also takes all real values. For the actual orbit we take v on the upper side of the rectangle, from $v = \omega_1 + \omega_2$, $p = \alpha(1 + e)$, corresponding to perihelion, to $v = \omega_2$, $p = \alpha(1 - e)$ or aphelion.

Finally, the corresponding change in φ is $\varphi = \sqrt{2}\omega_1$, where

$$\left(\frac{\omega_1}{\pi}\right)^4 g_2 = \tfrac{1}{12}$$

for q^2, being of order α^2, is here negligible. But from (3)

$$g_2 = \tfrac{1}{3}(1 - 12\alpha).$$

Hence

$$\frac{1}{4}\left(\frac{\varphi}{\pi}\right)^4 \cdot \frac{1}{3}(1 - 12\alpha) = \frac{1}{12}.$$

$$\left(\frac{\varphi}{\pi}\right)^4 = 1/(1 - 12\alpha) = 1 + 12\alpha,$$

$$\frac{\varphi}{\pi} = 1 + 3\alpha.$$

So that in passing from perihelion to aphelion the angle passed through is not π, but

$$\pi(1 + 3m^2/h^2).$$

A ONE-TO-ONE REPRESENTATION OF GEODESICS ON A SURFACE OF NEGATIVE CURVATURE.

By Harold Marston Morse.

Introduction.

Surfaces of negative curvature and geodesics upon such surfaces have been considered by J. Hadamard in the paper,* cited below. The paper by Hadamard is in two parts, in the first of which he establishes the existence of a very general class of surfaces of negative curvature; the remainder of Hadamard's paper is devoted to considerations of geodesics upon such surfaces.

In the present paper only those geodesics on the given surfaces of negative curvature are considered which, if continued indefinitely in either sense, lie wholly in a finite portion of space. A class of curves is introduced each of which consists of an unending succession of the curve segments by which the given surface, when rendered simply connected, is bounded. It is shown how a curve of this class can be chosen so as to uniquely characterize some geodesic lying wholly in a finite portion of space. Conversely, it is shown that every geodesic lying wholly in a finite portion of space, is uniquely characterized by some curve of the above class. Use is made of this representation of the geodesics considered to prove several fundamental theorems.

The results of this paper and the representation of geodesics obtained here, will be used in a later paper to establish the existence of a class of geodesics called recurrent geodesics of the discontinuous type.† This class of geodesics offers the first proof that has been given in the general theory of dynamical systems, of the existence of recurrent motions of the discontinuous type.

Part I.

The Surface.

§ 1. We will consider surfaces without singularities in finite space. We will suppose the surface divisible into overlapping regions such that every point of the surface lying in a finite portion of space is contained as an interior point in some one of a finite number of these regions, and such that

* Les surfaces à courbures opposées et leur lignes geodesiques," Liouville, *Journal de Mathématique*, 5 Sér., t. 4, p. 27, 1898.

† G. D. Birkhoff, "Quelques théorèmes sur le mouvement des systèmes dynamiques," *Bull. de la Soc. Math. de France*, Vol. 40, p. 303, 1912.

the Cartesian coördinates x, y, z, of the points of any one of these regions can be expressed in terms of two parameters, u and v, by means of functions with continuous derivatives up to a convenient order, at least the third, and such that

$$\left(\frac{D(xy)}{D(uv)}\right)^2+\left(\frac{D(xz)}{D(uv)}\right)^2+\left(\frac{D(yz)}{D(uv)}\right)^2 \neq 0.$$

By a *curve* on the surface we will understand any set of points on the surface in continuous correspondence with the points of an interval on a straight line, including one, both, or neither of its end points.

We will suppose the Gaussian curvature of the surface to be negative at every point, with the possible exception of a finite number of points, at which points the curvature will necessarily be zero. A first result, given by Hadamard in the paper already referred to, is that a surface of negative curvature cannot be contained in any finite portion of space.

§ 2. By a *funnel* of a surface will be meant a portion of a surface topographically equivalent to either one of the two surfaces obtained by cutting an unbounded circular cylinder by a plane perpendicular to its axis. We will consider surfaces of negative curvature whose points outside of a sufficiently large sphere with center at the origin consist of a finite number of funnels. Each of these funnels will be cut off from the rest of the surface along a simple closed curve. These curves will be taken sufficiently remote on the funnels to be entirely distinct from one another.

An unparted hyperboloid of revolution is an example of a surface of negative curvature with two funnels.

From the definition of a funnel, it follows that by a continuous deformation of the closed curve forming the boundary of the funnel, the funnel may be swept out in such a way that every point of the funnel is reached once and only once. Hadamard considers two classes of funnels: those which can be swept out by closed curves which remain less in length than some fixed quantity, and those which do not possess this property. Surfaces with funnels of the first sort are for several reasons of less general interest than those with funnels of the second sort. In the present paper surfaces with funnels only of the second sort will be considered. Hadamard showed that there exist surfaces of negative curvature possessing funnels all of the second sort of any arbitrary number exceeding one, and such that the surface obtained by cutting off these funnels is of an arbitrary genus.

§ 3. We shall consider surfaces which possess at least two funnels of the second sort, and of the surfaces with just two funnels of the second sort we will exclude those surfaces that are topographically equivalent to an unbounded circular cylinder. Hadamard proves that on the surfaces we

are considering there exists one and only one closed geodesic that is deformable into the boundary of a given funnel, and that this geodesic possesses no multiple points, and no points in common with the other closed geodesics that are deformable into the boundaries of the other funnels.

We shall denote these closed geodesics, say v in number, by

$$(1) \qquad g_1, g_2, \cdots, g_v.$$

They will form the complete boundary of a part of the surface, contained in a finite part of space. We denote this bounded surface by S. It may be proved as a consequence of the assumptions concerning the surface, made in § 1, that S is of finite connectivity. According to the well known theory, S may be rendered simply connected as follows: We first cut S along a system of curves,

$$h_1, h_2, \cdots, h_{v-1},$$

each of which has one end point on an arbitrarily chosen point, P on g_v, and the other, respectively, on the geodesic,

$$g_1, g_2, \cdots, g_{v-1}$$

with the same subscript; and no two of which have a point other than P in common, and no points other than their end points in common with the geodesics of the set (1). There then results a surface with a single boundary. This surface can accordingly be rendered simply connected by $2p$ curves,

$$c_1, c_2, \cdots, c_{2p},$$

each of which curves can be taken as beginning and ending at P, and having no other points than P in common with any of the other curves or geodesics.

We denote by T, the simply connected piece of surface obtained by cutting S along the above curves. It may readily be proved as a consequence of the assumptions made concerning the representation of the given surface, that T is topographically equivalent to a plane region consisting of the interior and boundary points of a circle.

The Infinitely Leaved Surface M.

§ 4. We suppose an infinite number of distinct copies of T spread out above the surface S, in such a manner that points arising from a given point of S overhang the given point of S. We denote by

$$A_i^+, \qquad i = 1, 2, \cdots, 2p + v - 1.$$

that boundary piece of T which arises from an arbitrarily chosen side of the ith one of the $2p + v - 1$ cuts,

$$(1) \qquad c_1, c_2, \cdots, c_{2p}; \qquad h_1, h_2, \cdots, h_{v-1},$$

which we have made in S, and by A_i^- that boundary piece of T that arises from the opposite side of the same cut.

Consider now a first copy of T, say M_1, and a particular boundary piece of M_1, say A_i^+. We join M_1 to a second copy of T, joining A_i^+ of M_1 to A_i^- of the second copy of T. In the same manner, we join to M_1 at each different piece of the remaining boundary pieces, a different copy of T, joining each two copies of T along two boundary pieces arising from opposite sides of the same cut. We denote the bounded surface so obtained by M_2.

Proceeding in the same manner, we join to each different boundary piece of M_2 a different copy of T, and one not already a part of M_2, obtaining thereby a new surface M_3. From M_3 we form a new surface M_4, as we formed M_3 from M_2, and continue this process indefinitely, obtaining an enumerable infinity of surfaces each of which contains the preceding. We will understand two different copies of T making up M_n, to have a point in common, if and only if we have expressly joined the two copies together at this point.

§ 5. DEFINITION. *By the surface M we will understand that infinitely leaved surface spread over S that contains each of the surfaces M_n, and every point of which is contained in some one of the surfaces M_n.*

The boundaries of M will consist of those boundary pieces, of the copies of T that make M up, that arise from the geodesics,

$$g_1, g_2, \cdots, g_v.$$

We have made the cuts in S so that on each of the boundary pieces,

$$g_1, g_2, \cdots, g_{v-1}, \tag{2}$$

lies one end point of just one cut. The remaining $4p + v - 1$ end points of the cuts lie at a single point on g_v. On M a sufficiently restricted neighborhood of any one of the points on the geodesics (2), at which lies an end point of a cut, is, when severed along this cut, best described as topographically equivalent to the neighborhood of a point on the boundary of a half plane, when that half plane is severed along a ray distinct from the boundary of the half plane, and issuing from the given point. Similarly a sufficiently restricted neighborhood of that point on g_v at which lie the remaining $4p + v - 1$ end points of cuts, is, when severed along these cuts, topographically equivalent to the neighborhood of a point on the boundary of a half plane, when that half plane is severed along $4p + v - 1$ rays, distinct from each other and from the boundary of the half plane, and issuing from the given point. A sufficiently restricted neighborhood of any other point of M, is topographically equivalent to the complete neighborhood of any point in the plane.

We may conclude that *a sufficiently restricted neighborhood of any point on M is a one-leaved copy of the neighborhood on S of the point overhung on S.*

§ 6. It will be convenient to denote by R a plane region consisting of the points on and interior to a circle.

Any single copy of T is topographically equivalent to a region R. From the method of formation of the surfaces M_n of § 4, it follows that any one of the surfaces M_n is topographically equivalent to a region R. Now from the definition of M in § 5, it follows that any set of points on M that lie in only a finite number of copies of T making up M, will be contained in some one of the surfaces M_n forming a part of M. *Hence any set of points that lie in only a finite number of the copies of T that make up M, will be contained in some region of M, topographically equivalent to a region R.*

§ 7. An infinite set of points on M will be said to have a *limit point* P, if there exists an infinite number of points of the set in every neighborhood of P, on M.

We will prove that a set of points every infinite subset of which has at least one limit point on M, cannot have points in more than a finite number of different copies of T making up M.

If there were points of the set in more than a finite number of the copies of T, there would exist a subset of the given set, say Q, containing an infinite number of points of the given set, and such that each different point would lie in a different copy of T. By the hypothesis made concerning the given set, Q has at least one limit point on M, say P. There are accordingly an infinite number of points of Q in every neighborhood of P, and since no two points of Q lie in the same copy of T, there must be an infinite number of copies of T with points in the neighborhood of P. This is impossible; for all the points in a sufficiently restricted neighborhood of a point on M lie on at most $4p + v$ copies of T, that are joined together at that point.

Hence the result required is proved. Combining this result with the conclusion of the preceding section, we have that *a set of points on M every infinite subset of which has at least one limit point on M, can be included in a region of M, made up of a finite number of copies of T, and topographically equivalent to the plane region consisting of the interior and boundary points of a circle.*

Curves on M Overhanging Curves on S.

§ 8. DEFINITION. Let there be given on S a curve k, and on k a point P. Let P' be any point on M, overhanging P. Starting from a particular point of k, say Q, let P trace out k, first in one sense, and then starting again at Q, in the other sense. As P moves continuously on S, let P' move continuously on M, always overhanging P, and starting at both the times that P starts at Q, at the same point, a point, say Q', overhanging Q. The

continuous curve traced out by P' on M, will be said to *correspond* to k on S.

It follows from the result of § 5, that the curve traced out by P', is uniquely determined by k and the choice of the points Q and Q'.

DEFINITION. Let h be a curve on S that is closed. Let h be cut at some point so as to form a curve segment k. Let h' be any curve segment on M corresponding, in the sense of the preceding definition, to k. h and h' will be said to *correspond.*

Every curve on M obviously corresponds to one and only one curve on S, while a given curve on S will correspond to an infinite number of different curves on M.

§ 9. *If a curve on S is both closed and deformable into a point, it corresponds on M only to curves that are closed.*

Let h be a curve on S that is both closed and deformable into a point. Let Q' and P' be the end points of a curve segment on M corresponding in the sense of the preceding definition, to h. Since h is closed, Q' and P' overhang a single point on h, say Q.

Suppose now that the above lemma were not true, and that Q' and P' were distinct points on M. Let h be continuously deformed into a point. At the same time let the curve segment joining Q' to P', say h', be continuously deformed on M in such a manner that h' corresponds at each stage of the deformation to the curve into which h has been deformed at that stage, while Q' and P' overhang Q. When, at the end of this process, h reduces to a point, h' must also reduce to a point.

On the other hand P' and Q', the end points of h', in their initial positions were supposed to be distinct points. Since P' and Q' move continuously throughout the deformation, there must then be a first position on M, say P'', at which they are coincident. P' and Q', throughout this deformation, overhung one point of S, namely Q. Hence the point P'' will have in every neighborhood on M pairs of points overhanging the same point of S. This, however, is contrary to the result of § 5, from which contradiction we infer the truth of the above lemma.

§ 10. The following lemma serves as the converse of the preceding.

If a curve on M is closed, it corresponds on S to a curve that is both closed and deformable into a point.

Denote by h the given closed curve on M. It follows from § 7 that there exists a region of M, say M', that contains all the points of h as interior points, and is topographically equivalent to a plane region consisting of the interior and boundary points of a circle. On M', h can be deformed into a point, and hence on M.

Denote by F a continuous family of closed curves on M through the mediation of which h may be deformed into a point on M. On S the

closed curve corresponding to h, may be deformed into a point through the mediation of that continuous family of closed curves which correspond to the family F on M.

§ 11. DEFINITION. Let r be any integer, positive, negative, or zero. Let T_r denote a particular copy of T of M. By a *linear set* of copies of T of M, will be understood a region of M consisting of a set of the copies of T in M of the form,

$$\cdots\ T_{-2},\ T_{-1},\ T_0,\ T_1,\ T_2,\ \cdots, \tag{1}$$

or of the form of any subset of consecutive symbols of (1), in which any three successive copies of T are three different copies of T of M, and in which any two successive copies of T are copies of T that in M are joined along a common boundary piece. A linear set which has no first or last copy of T, will be termed an *unending* linear set.

A linear set is of the nature of an unclosed ribbon of surface. That a linear set does not form a multiple-covered region of M will now be proved.

(*A*) *No copy of T of M appears more than once in any linear set.*

Starting with any copy of T, say T^0, of a given linear set, if possible, let T' be the first successor of T^0 that is identical with some copy of T of the given linear set between T^0 and T', say T''. That linear subset of the given linear set that consists of T'' and its successors up to, but not including T', forms a simply covered, connected region of M, in which the two sides of any boundary piece common to two of its successive copies of T, can be joined by a curve that does not cross that boundary piece. This is impossible, since each boundary piece common to two copies of T of M joins two boundary points of M and divides M into two regions. Cf. § 5 and § 6.

(*B*) *If T' and T'' are any two copies of T of the copies of T that make up M, there exists one and only one linear set of which T' is the first, and T'' the last member.*

We form a linear set in which the first copy of T is T' and in which the successor of T', say T_1, is that copy of T that adjoins T along that boundary piece of T' that separates T' from that part of the surface M that contains T''. If T_1 is not T'', its successor shall be that copy of T that adjoins T_1 along that boundary piece of T_1 that separates T_1 from that part of the surface in which T'' lies. We continue this process indefinitely, or until T'' appears in the linear set formed.

Join any interior point of T' to an interior point of T'' by a curve, say k, lying wholly on M. Each copy of T determined in the process of the preceding paragraph contains at least a point of k. But from the result of § 7 we infer that there are only a finite number of copies of T of M that contain any point of k. It follows that the process of the preceding paragraph will lead after a finite number of steps to T''.

There is but one linear set of which T' is the first and T'' the last copy of T. For no such linear set can contain as a part of its boundary that boundary piece of T' that separates T' from that part of the surface that contains T''. Hence, in any such linear set T' and T_1 must appear at least once as successive copies of T. But according to the result (A), no copy of T of M appears more than once in any linear set; it follows that T_1 is the successor of T' in any linear set joining T' to T''.

In a similar manner it follows that any linear set joining T' to T'', contains T_2 as the successor to T_1. Continuing this method of proof, it is seen that every linear set joining T' to T'', is identical with the first such linear set that we have formed.

§ 12. We denote by H, the set of all curve segments on M each of which corresponds on the original surface to some one of the segments,

$$c_1, c_2, \cdots, c_{2p}; \qquad h_1, h_2, \cdots, h_{v-1}; \qquad g_1, g_2, \cdots, g_v.$$

Every boundary piece of the copies of T of M, is included in the set H.

DEFINITION. Let r be any integer, positive, negative, or zero. Let k_r be a particular piece of the set H of M. By a *reduced curve* of the set H will be understood a continuous curve composed of a set of pieces of the set H of the form,

$$\cdots, k_{-2}, k_{-1}, k_0, k_1, k_2, \cdots, \tag{1}$$

or of the form of any subset of consecutive symbols of (1), in which no two consecutive pieces are copies of the same piece of the set H, and which contains no pieces corresponding to g_v on the original surface. A reduced curve without end points will be called an *unending* reduced curve.

(A) *No reduced curve can begin and end at the same point.*

Starting with any point, say P, of a given reduced curve, if possible, let P' be the first point following P that is identical with some point of the reduced curve between P and P', say P''. The segment $P'P''$ of the given reduced curve forms a closed curve without multiple points, which closed curve we denote by C.

According to the result of section 7, C can be included in some simply connected region of M consisting of a finite number of copies of T. C, accordingly, forms the boundary of a simply connected region of M, say R, all of whose points are contained in a finite number of copies of T of M. Since C contains no points interior to any copy of T, R contains each copy of T of which it contains any interior points.

Taking the copies of T in the order in which they were added in section 4 to form M, let T' be the last copy of T of R. T', like every other copy of T of M, is joined along a common boundary piece to but one copy of T of

M, say T'', that was added to M prior to T'. R accordingly contains none of the copies of T that are adjoined to T' along a common boundary piece except T''. Hence all of the boundary of T', except the piece common to T' and T'', must form part of the boundary of R, and hence a part of C. In particular, C contains that piece of the boundary of T' that corresponds on the original surface to the boundary geodesic g_v, contrary to the definition of a reduced curve. From this contradiction we infer the truth of the lemma.

The following result is an immediate consequence of the result (A).

(B) *A reduced curve can have no multiple points.*

(C) *There is one and only one reduced curve joining any two given points on pieces of the set H.*

Denote the two given points by P and Q. Among the curves consisting of any continuous succession of pieces of the set H, there obviously exists a variety of curves joining P to Q. If in any such curve, each piece that corresponds on the original surface to g_v, be replaced by the remainder of the boundary of the copy of T containing that piece, there will then result a curve containing no piece corresponding to g_v on the original surface. Of all curves of the latter class joining P to Q, any one that contains the minimum number of pieces of the set H, will be such that no two successive pieces are copies of the same piece of the set H, and will accordingly be a reduced curve.

If possible, let h' and h'' be two different reduced curves joining P to Q. Tracing out h' and h'' respectively, starting from P, let k' and k'' be the first pieces on h' and h'' respectively, that are different. That portion of h' that begins at Q, and ends with k', followed by that portion of h'' that begins with k'', and ends with Q, is a reduced curve with initial and final point at the same point, namely Q. This is contrary to the result (A) of this section, from which contradiction we infer the truth of the lemma (C).

(D) *A reduced curve never returns to a copy of T which it has once left.*

If this assertion be false, there exists some continuous segment of a reduced curve, say k, with end points, say P and Q, belonging to the same copy of T of M, and with no further points in common with that copy of T. A reduced curve cannot begin and end at the same point; hence P and Q are different points. However, let k' be that part of the boundary of the given copy of T that joins P to Q, and does not contain any piece of the set H corresponding to g_v on the original surface. k', taken with k, forms a segment of a reduced curve that may be considered as beginning and ending at P. We conclude that no such segment as k exists, and the lemma is proved.

§ 13. The following is the first of two theorems showing the relation of reduced curves to linear sets.

(*A*) *An unending linear set of copies of T contains one and only one unending reduced curve.*

Let T^0, T' and T'', be any three successive copies of T in the given linear set. There exist two distinct and continuous portions of the boundary of T', of which one may reduce to a point such that each joins some boundary point of T^0 to a boundary point of T'', without containing any other points of T^0 and T''. We associate with T' that one of these portions that does not contain a piece of the set H corresponding to g_v on the original surface. Denote this portion by k. The set of all curve segments such as k, taken in the order in which they arise from the given linear set, will not necessarily form a continuous curve. However by a proper addition of the boundary pieces common to successive copies of T of the given linear set, the resulting curve will be continuous. If these additions are made only when necessary to make the curve continuous, the resulting curve will also be a reduced curve.

If possible, let r be a second reduced curve contained in the given linear set. As proved in the preceding section, a reduced curve never returns to a copy of T which it has once left; it follows that r contains at least a point of each of the pieces of the set H that separate members of the given linear set, and hence contains each of the points or segments k, associated in the first paragraph of this proof with each copy of T of the given linear set. But according to the result (C) of section 12, there is but one segment of a reduced curve joining any two points of M. Hence any reduced curve that contains each of the points or curve segments k, associated with each copy of T of the given linear set, is thereby uniquely determined. There is thus but one reduced curve contained in the given linear set.

The preceding result has its converse in the following:

(*B*) *An unending reduced curve is contained in one and only one linear set, which set is an unending linear set.*

Let

$$\cdots, P_{-2}, P_{-1}, P_0, P_1, P_2, \cdots,$$

be a set of points which lie on the given reduced curve in the order of their subscripts, and which are such that any point of the given curve lies between P_{-n} and P_n for n a sufficiently large positive integer, and where P_{-1} and P_1 are supposed to be taken so as not to lie on a common copy of T of M. Let C_n be the segment joining P_{-n} to P_n of the given reduced curve. Let L_n be a linear set containing P_{-n} and P_n, and made up of the smallest possible number of copies of T.

C_n lies wholly in L_n. If this assertion be false, let k be a piece of the set H which forms part of the boundary of L_n, and contains a point at which C_n leaves L_n. The end points of C_n lie on L_n; hence C_n returns

to L_n. But k divides the whole of M into two regions; hence to return to L_n, C_n would have to return to k, which is impossible, since a reduced curve never returns to a copy of T which it has once left.

We will now prove that if r is any positive integer greater than n, the linear set L_r contains L_n. Let h be any one of the boundary pieces common to two adjacent copies of T of L_n. C_n contains points, not on h, in each of the two regions into which M is divided by h, for otherwise one of the two linear sets into which L_n is divided by h, would contain all the points of C_n and be a linear set containing fewer copies of T than does L_n, contrary to the hypothesis concerning L_n. Hence any connected region that contains all the points of C_n, must contain the two copies of T of L_n that are adjoined along h. Hence any connected region that contains all of the points of C_n, must contain each copy of T of L_n. In particular, L_r contains all the points of C_n, and hence contains L_n.

The set of all copies of T of all regions L_n, for all positive integers n, will accordingly constitute a linear set, which we denote by L. Any point of the given curve is contained on every curve C_n for sufficiently large values of n, and hence is contained in L. We will now prove that L is an unending linear set.

As seen earlier in the proof, the two linear sets into which L_n and hence L is divided by any boundary piece h, common to copies of T of L, each contain points of the given reduced curve that are not on h. Since a reduced curve never returns to a copy of T which it has once left, the removal of h divides the given reduced curve into just two continuous portions, which lie respectively in the two linear sets into which L is divided by h. But each of these continuous portions of the given reduced curve will contain an infinite number of different pieces of the set H, so that each of the two linear sets into which L is divided by h, must contain an infinite number of different copies of T. We conclude that L is an unending linear set.

Any second linear set that contains L, would have to contain each of the linear sets L_n, hence every copy of T of L. Since there is but one linear set joining any two copies of T of M, any linear set containing all the copies of T of L, is identical with L.

The two results of this section may be combined in the following fundamental theorem:

THEOREM I. *There is a one to one correspondence between the set of all unending reduced curves, and the set of all unending linear sets of M, in which each reduced curve corresponds to that linear set in which it is contained.*

PART II.

Some General Properties of Geodesics on M.

§ 14. Hadamard proved that there is on S one and only one geodesic joining two given points, and deformable into an arbitrary curve joining the two given points, and that this geodesic is shorter than any other rectifiable curve deformable into the given curve joining the two given points. With the aid of the results of sections 9 and 10, Hadamard's result becomes the following theorem.

THEOREM II. *There exists on M one and only one geodesic joining two given points, and this geodesic is shorter than any other rectifiable curve joining the two given points.*

COROLLARY I. *On M two geodesics can intersect in but one point.*

COROLLARY II. *On M a geodesic can have no multiple points.*

§ 15. On a surface representable in the manner in which the given surface is representable, there exists one and only one geodesic through a given point, and tangent to a given direction.

DEFINITION. A point on the surface, and a direction tangent to the surface will be called an *element*, and will be said to *define that sensed geodesic* that passes through the initial point of the given element, and is such that its positive tangent direction at that point agrees with the direction of the given element.

If u and v are parameters in any representation of a part of the surface, and if θ' is the angle which a given tangent direction makes at the point $(u'v')$ with the positive tangent to the curve $u = u'$, then $(u'v'\theta')$ will represent an element of the given surface. We shall understand by each statement of metric relations between elements, the same statement of metric relations between the points in space of three dimensions, obtained by considering the complex $(u'v'\theta')$ as the Cartesian coördinates of a point.

§ 16. Let G be any geodesic segment lying on the original uncut surface. G is an extremal in the Calculus of Variations problem of minimizing the arc length, from which theory we can readily obtain the following theorem that describes the nature of the variation of G with variation of its initial element.*

Corresponding to any positive constants, e and h, there exists a positive constant d, so small, that if any two elements, with initial points on the bounded surface S, lie within d of each other, and if a second pair of elements lie respectively on the two geodesics defined by the first two elements, and if further the initial points of this second pair of elements lie respectively at a distance, measured along the given geodesics from the geodesics' initial

* Cf. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 219.

points, that is the same in both cases and that does not exceed h, the second pair of elements will lie within e of each other.

§ 17. According to Theorem II, § 14, any given geodesic on M is shorter than any other rectifiable curve joining its end points. According to the theory of the Calculus of Variations, this is a case where there exists on the given geodesic on M no point conjugate to a given point on the given geodesic. A particular consequence* of this result, as given in the theory of the Calculus of Variations, is that, if we vary the end points of a given geodesic segment, there exists further geodesic segments joining these end points and varying continuously with the end points, both in position and in length. We have seen that there is no more than one geodesic segment joining any two points on M. Whence we may say that the length of the geodesic segment, joining two points on the surface M, is a single-valued continuous function of the position of these points. In particular we have the result:

There exists a finite upper limit to the lengths of all geodesic segments joining pairs of points lying on a closed set of points on M.

§ 18. The following statement, of importance in the developments that are to follow, may readily be proved as a consequence of the development given by Hadamard.

Corresponding to two arbitrary positive constants, e and d, there exists a positive constant h, so large, that if on M each point of a first geodesic segment G, of length h, lies within a geodesic distance d of some point of a second geodesic segment, then this second geodesic segment has at least one element within an e of that element of G which lies at the mid point of G.

§ 19. *The Network H Replaced by a Network H' of Geodesic Segments.* We denote by H', the set of all geodesic segments on M each of which joins two end points of a piece of the set H, defined in § 12.

We shall prove the following lemma:

Two pieces of the set H' arising from two different pieces of H, can have no points in common other than one end point.

Let P and Q, P' and Q', be respectively the end points of two different pieces of the set H. Denote by h the geodesic segment that joins P to Q, and by h' the geodesic segment that joins P' to Q'. All cases are included in the three following cases:

CASE I. The two given pieces of the set H have an end point in common. In particular suppose $P = P'$; if $P = P'$, it follows from the nature of the construction of the surface M, that $Q \neq Q'$. Hence, if $P = P'$, the geodesic segments, h and h', have P as a common end point, and are not identical, since $Q \neq Q'$. With the aid of Corollary II, § 14, we conclude that h and h' can have no other point than P in common.

* Cf. Bolza, loc. cit., p. 307.

CASE II. $P \neq P'$, and $Q \neq Q'$, and one or both of the geodesic segments, h and h', form a piece of the geodesic boundary of the surface M.

h and h' are not identical, since $P \neq P'$, and $Q \neq Q'$. In case h were a piece of the geodesic boundary of M, h' could not meet h in other than an end point without passing off from the surface at that point, contrary to the result of Theorem II, § 14.

CASE III. $P \neq P'$, and $Q \neq Q'$, while neither h nor h' form a piece of the geodesic boundary of the surface M.

According to the result of § 7, there exists on M, a simply connected region, bounded by a simple closed curve, and containing the two given pieces of the set H and the geodesic segments h and h'. This region, which we denote by R, we will suppose taken so large that no points of the two given pieces of the set H, nor of the geodesic segments h and h', are boundary points of R unless they are boundary points of M.

P, Q, P', *and* Q', all lie on the boundary of M, and hence on the boundary of R. Since $P \neq P'$, and $Q \neq Q'$, the two given pieces of the set H have no points in common; it follows that P, Q, P', and Q', lie on the boundary of R in the circular order named.

On the other hand, it is an hypothesis of this case, that neither h nor h' are a piece of the geodesic boundary of M; it follows that neither h nor h' have points other than their end points on the boundary of M, and hence of R. A particular consequence is that h divides R into two regions. If now h' crossed h, its end point P' would lie in one of the two regions into which R is divided by h, while its remaining end point, Q', would lie in the other such region. P, P', Q, Q', would thus lie on the boundary of R in the circular order named. From this contradiction we infer the truth of the lemma in this case. The proof is thus given in general.

§ 20. We seek now to render the original surface simply connected by means of cuts made along geodesics.

In accordance with the result given in § 14, the end points of each one of the cuts by means of which the original surface was rendered simply connected, can be joined on the original surface by a geodesic segment deformable into the given cut. With the aid of the result of § 9, it appears that these geodesic segments correspond on M to the geodesic segments of H', and further that two of these geodesic segments arising from different cuts correspond on M only to different members of the set H'. From the result of the preceding section we conclude that no two of these geodesic segments have any points in common other than their end points.

If now the original surface be cut along these geodesic segments, the resulting surface, which we denote by U, will be simply connected. The infinitely leaved surface M can be formed by joining together copies of U in the same manner as it was formed from copies of T.

We define a reduced curve of the set H', and a linear set of copies of U, by replacing H and T in the definitions of sections 11 and 12, by H' and U, respectively. In the same manner, we obtain from the results of sections 11 and 13, the following results in terms of H' and U.

(*A*) *If U' and U'' are any two copies of U of the copies of U that make up M, there exists one and only one linear set of which U' is the first and U'' the last member.*

(*B*) *There is a one to one correspondence between the set of all unending reduced curves of the set H', and the set of all unending linear sets of copies of U, in which each reduced curve corresponds to that linear set in which it is contained.*

Geodesics Lying Wholly on M.

§ 21. Let G be a geodesic lying wholly on M. G cannot become infinite in length in any one copy of U, as follows from the result of section 17. In leaving a copy of U of M, G cannot be tangent to any one of the geodesic segments that separate that copy of U from the remainder of M. For in such a case G would coincide with that geodesic segment, and pass off from M at its end points. Further, it follows from Corollary I, section 14, that G can have but one point of intersection with any of the geodesic segments that separate the different copies of U of M. Hence G never returns to a copy of U of M which it has once left. Since G cannot become infinite in length in any one copy of U, it follows that either of the two portions into which G may be divided by an arbitrary one of its points, has points in common with an infinite number of different copies of U.

From these last two results, it follows by a proof, which except for terminology, may be given as a repetition of the proof of (*B*), section 13, that *a geodesic lying wholly on M, is contained in one and only one linear set, which set must be an unending linear set.*

§ 22. As a converse to the preceding result, we have the following:

If there be given any unending linear set, there exists one, and only one geodesic contained wholly in the given linear set, and this geodesic has at least one point in each copy of U of the given linear set.

Let an unending linear set of copies of U be given as follows:

$$\cdots,\ U_{-2},\ U_{-1},\ U_0,\ U_1,\ U_2,\ \cdots.$$

Let g_n be a geodesic segment joining any interior point of U_{-n} to an interior point of U_n. The set of copies of U which g_n passes through, is seen to form a linear set joining U_{-n} to U_n. But from the result of section 11, it follows that there is but one such linear set joining U_{-n} to U_n. The linear set,

$$U_{-n},\ U_{-n+1},\ \cdots,\ U_0,\ \cdots,\ U_{n-1}U_n,$$

is one such set; we conclude that this is the set of copies of U through which g_n passes.

Denote by E_n an element on that part of g_n that lies in U_0. The set of all elements E_n will have a limit element. Let G be the geodesic defined by this element. We will first show that G has a point in each copy of U of the given linear set.

Let r be any positive integer. For integers $n > r$ that portion of g_n in

$$(1) \qquad U_{-r}, U_{-r+1}, \cdots, U_0, \cdots, U_{r-1}U_r,$$

is according to the result of section 17, less in length than some fixed quantity independent of n. Now a finite segment of a geodesic varies continuously with its initial element. It follows that G possesses a finite segment, say G_r, which has a point in each copy of U of (1), and which is wholly contained in the set (1). From the fact that G_r has a point in each copy of U of (1), we may conclude that G has a point in each copy of the given linear set.

We seek to prove that G is wholly contained in the given linear set. Any finite segment of any geodesic on M has points in not more than a finite number of copies of U. Cf. section 7. We may conclude that, for r sufficiently large, any given segment of G that begins with a point of U_0, is included in one of the two portions into which G_r is divided by that point. Thus every point of G lies on some segment G_r. But every point of G_r, and hence every point of G lies in the given linear set.

If there were a second geodesic, say G', contained in the same linear set as G, it would follow from the result of section 18, that every element of G would be a limit element of elements on G'. This is impossible, since G' can never return to a copy of U which it has once left.

The results of this section and the preceding, are summed up in the following:

THEOREM III. *There is a one-to-one correspondence between the set of all geodesics lying wholly on M, and the set of all unending linear sets, in which each geodesic corresponds to that linear set in which it is contained.*

A similar theorem is now obtained from (B) in § 20.

THEOREM IV. *There is a one-to-one correspondence between the set of all geodesics lying wholly on M, and the set of all reduced curves of the set H', in which each geodesic corresponds to that reduced curve that is contained in the same linear set.*

Congruent Curves.

§ 23. DEFINITION. Two curves on M that correspond to the same curve on S will be said to be *congruent.*

The first statement of the following theorem serves as an existence proof for closed geodesics on the original surface.

(*A*) *If a reduced curve on M consists of successive mutually congruent portions, the geodesic that lies in the same linear set as the given reduced curve, consists also of successive mutually congruent portions. Conversely, if a geodesic lying wholly on M consists of successive mutually congruent portions, the reduced curve that lies in the same linear set, consists also of successive mutually congruent portions.*

Suppose the given reduced curve consists of successive mutually congruent portions. Let the configuration, consisting of the given reduced curve and the geodesic lying in the same linear set, be carried as a whole into that congruent configuration in which one of the successive mutually congruent portions of the given reduced curve is carried into the succeeding congruent portion. Denote this transformation by T.

The given geodesic is carried into itself by T. For otherwise there would exist two different geodesics, namely the given geodesic and the geodesic into which it is carried by T, both lying in the same linear set as the given reduced curve. This is contrary to the result of Theorem IV, section 22.

Now let P' be any point on the given geodesic. Let P'' be the point in which P' is carried by T. P'' lies on the given geodesic, since the given geodesic is carried into itself by T. The successive images of the geodesic segment $P'P''$ which result through repetition of T and its inverse, will be successive mutually congruent portions of the given geodesic, all of whose points can be reached by a sufficient number of such transformations.

The converse statement of the theorem can be proved in a similar manner.

In the same manner also, the following can be established.

(*B*) *Let there be given a first reduced curve and geodesic contained in a single linear set, and a second reduced curve and geodesic that are also contained in a single linear set. If the two given reduced curves are congruent, the two geodesics must also be congruent; and conversely, if the two given geodesics are congruent, the two reduced curves must also be congruent.*

Variation of Geodesics with Their Initial Elements.

§ 24. The following statement depends upon the result given in section 18.

(*A*) Corresponding to any positive quantity e, there exists a positive integer n, so large, that if any two unending linear sets have in common a region U, say U', together with the first n regions U succeeding U' in either sense in the given linear set, then there exists in U', and on each of the two geodesics that pass respectively through the two given linear sets, at least one element that lies within e of some element on that part of the other geodesic that lies in U'.

The converse statement depends upon the property of continuous variation of a geodesic with its initial element, as given in section 16.

(B) Corresponding to any positive integer n, there exists a positive constant e, so small, that if on each of two geodesics there exists some element within e of some element on the other, then the two linear sets through which the two given geodesics respectively pass, possess in common that linear set that contains the region or regions U, in which the two given elements lie, together with n regions U succeeding these regions in either sense.

By virtue of the relations between unending reduced curves and corresponding unending linear sets, as given in the proofs of section 13, the statements of (A) and (B) of this section become the following:

THEOREM VI. *Corresponding to any positive constant e, there exists a positive constant k, so large, that if two unending reduced curves possess in common a continuous segment of length exceeding k, the two corresponding geodesics each have at least one element within e of some element on the other, and with initial point in the same copy of U as the mid point of the common reduced segment. Conversely, corresponding to any positive constant k, there exists a positive constant d, so small, that if on each of two geodesics there exists some element within d of some element on the other, the two corresponding reduced curves possess in common a segment of length k, with mid point in the same copy of U as the initial point of either of the two elements.*

§ 25. DEFINITION. Two elements E' and E'' of a set of elements E on M, will be said to be *mutually accessible* in E, if corresponding to any positive constant e, there exists in the set E a finite ordered subset of elements of which the first element is E', and the last element E'', while each element of the subset, excepting the last, lies within e of the following element.

THEOREM VI. *In the set of all elements on a closed region of M, and on geodesics lying wholly on M, no two elements lying on different geodesics are mutually accessible.*

Let G' and G'' be two different geodesics lying wholly on M. Let E' and E'' be two elements lying respectively on G' and G''. Let R be any closed region of M, in which the initial points of E' and E'' lie. We will show that E' and E'' are not mutually accessible in the set of all elements on R, and on geodesics lying wholly on M.

G' and G'' are different geodesics, and are accordingly not contained in the same linear set of copies of U. There therefore exists some geodesic segment, of the geodesic segments that separate different copies of U of M, that G' crosses and that G'' does not cross. Denote this geodesic segment by h. Denote by B' the set of all elements on R, and on geodesics that lie

wholly on M and cross h. Denote by B'' the set of all elements on R, and that lie on geodesics that lie wholly on M and do not cross h.

B' is a closed set. For it follows from the result of sections 16 and 17, that a limit element of elements of the set B', defines a geodesic, say G, lying wholly on M, and that is either tangent to h or else crosses h. But G cannot be tangent to h without passing off from M at the end points of h. Hence G crosses h. Thus B' is a closed set. In a similar manner it follows that B'' is a closed set.

From their definitions it follows that B' and B'' can have no elements in common. There accordingly exists a positive constant d, such that no element of B' lies within d of any element of B''. Hence E', which belongs to B', and E'', which belongs to B'', cannot be mutually accessible in the set of elements comprising the elements of B' and B''. But the elements of B' and B'' comprise all the elements on geodesics lying wholly on M, and with initial points in the region R.

The theorem thus is proved.

HARVARD UNIVERSITY,
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CONJUGATE SYSTEMS WITH INDETERMINATE AXIS CURVES.

By Ernest P. Lane.

I. Introduction.

The general projective theory of congruences may be studied* in connection with a completely integrable system of partial differential equations of the form

$$D) \qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= a\,y + b\,z + c\,y_u + d\,z_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v, \end{aligned}$$

where the subscripts indicate differentiations. In order that system (D) may be completely integrable, its coefficients, which are assumed to be analytic functions of the two arguments u and v, must satisfy the following integrability conditions:

$$c = f_u, \quad d' = f_v, \quad b = - d_v - df_v, \quad a' = - c_u - c'f_u, \quad mn - c'd = f_{uv},$$

$$\begin{aligned} m_{uu} + d_{vv} + df_{vv} + d_vf_v - f_um_u &= ma + db', \\ n_{vv} + c'_{uu} + c'f_{uu} + c'_uf_u - f_vn_v &= nb' + c'a, \\ 2m_un + mn_u &= a_v + f_umn + a'd, \\ m_vn + 2mn_v &= b'_u + f_vmn + bc', \end{aligned}$$

where f is an analytic function of u and v. Then system (D) has exactly four pairs of linearly independent solutions, $(y^{(i)}, z^{(i)})$, $(i = 1, \cdots 4)$, such that the general solution is of the form

$$y = \sum_{i=1}^{4} c^{(i)}y^{(i)}, \qquad z = \sum_{i=1}^{4} c^{(i)}z^{(i)},$$

where $c^{(1)}, \cdots, c^{(4)}$ are arbitrary constants. If $y^{(1)}, \cdots, y^{(4)}$ and $z^{(1)}, \cdots, z^{(4)}$ are interpreted as the homogeneous coördinates of two points P_y and P_z of three-dimensional space, these points will describe in general, for variable u and v, two surfaces S_y and S_z, but the possibility is not excluded that these surfaces reduce to curves. The line P_yP_z describes a congruence whose

* E. J. Wilczynski, "Sur la théorie générale des congruences." Mémoires publies par la classe des Sciences de l'Academie Royale de Belgique. Collection en 4°. Deuxième série. Tome III (1911). This paper will be cited hereafter as Brussels Paper.

focal surfaces are S_y and S_z, and whose developables are obtained by equating either u or v to a constant.

The form of system (D) is undisturbed by the group of transformations

$$(T) \qquad y = \lambda(u)\bar{y}, \qquad z = \mu(v)\bar{z}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

where λ, μ, α, β are arbitrary functions of the arguments indicated. Among the invariants of this group are the four coefficients m, n, c', d. It is easy to verify that I and l are also invariants, where

$$I = a + c\frac{m_u}{m} - \frac{m_{uu}}{m}, \qquad l = \frac{b}{d} + \frac{n_v}{n}.$$

Five fundamental covariants are y, z, ρ, σ, ϵ, where

$$\rho = y_u - \frac{m_u}{m}y, \qquad \sigma = z_v - \frac{n_v}{n}z, \qquad \epsilon = \rho_u - \frac{m_{1u}}{m_1}\rho,$$

and

$$m_1 = m\left(mn - \frac{\partial^2}{\partial u \partial v}\log m\right).$$

The axis of a point P on a surface has been defined,* with reference to a given conjugate system of curves on the surface, to be the line of intersection of the osculating planes at P of the two curves of the system which pass through P. The axis congruence of a surface is composed of the axes of all its points. An axis curve of a surface is a curve such that the axes of its points form a developable of the axis congruence.

The curves $u =$ const. and $v =$ const. form a conjugate system on S_y. For convenience of reference we quote here some results,† concerning the axis curves of S_y defined with reference to this conjugate system. The axis curves are given by the equation

$$(1) \qquad d_1\delta u^2 - mI\delta u\delta v - c'dm\delta v^2 = 0,$$

where

$$d_1 = d\left(c'd - \frac{\partial^2}{\partial u \partial v}\log d\right).$$

If the function τ be defined by

$$(2) \qquad \tau = lz + \sigma,$$

then the point P_τ is the point of intersection of the axis of P_y with the line P_zP_σ. The foci of the axis are determined by factoring the covariant

$$(C) \qquad c'd_1y^2 - mIy\tau - dm\tau^2.$$

* E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. XVI, No. 3, pp. 311–327. This paper will be cited hereafter as Congruences.

† Congruences, § 2.

It has been remarked* that the axis curves are indeterminate if

$$d_1 = mI = c'dm = 0. \tag{3}$$

It is the purpose of this paper to pursue the consequences of this remark and to study those surfaces upon which there exists a conjugate system with indeterminate axis curves.

The author takes advantage of this opportunity to express his apprecia- of the many helpful suggestions so kindly given him by Professor Wilczynski while this paper was in process of preparation.

I. FUNDAMENTAL THEOREM.

We shall now assume that the axis curves on S_y defined with reference to the parametric conjugate system, are indeterminate. Then the equations

$$d_1 = mI = c'dm = 0 \tag{3 bis}$$

are valid. If the surface S_y does not reduce to a curve, the invariant m does not vanish.† Similarly, if S_z is not degenerate, $n \neq 0$. Moreover, if S_y is not a developable surface, $d \neq 0$. We shall assume from now on that we are considering only those surfaces S_y which are nondegenerate and nondevelopable, and surfaces S_z which at least are nondegenerate. Under these assumptions we shall proceed to find formulas for the coefficients of system (D).

Equations (3) may be replaced by the equivalent equations

$$\frac{\partial^2}{\partial u \partial v} \log d = I = c' = 0. \tag{4}$$

As an immediate consequence we have

$$d = UV, \tag{5}$$

where U and V are nonvanishing functions of u alone and of v alone, repsectively.

When use is made of equations (4) and (5), the integrability conditions may be written in the form

$$c = f_u, \qquad d' = f_v, \qquad b = - UV' - UVf_v, \qquad a' = 0,$$

$$mn = f_{uv}, \qquad V'' + Vf_{vv} + V'f_v = Vb', \qquad n_{vv} - f_v n_v = nb'_v,$$

$$2m_u n + mn_u = a_v + f_u mn,$$

$$m_v n + 2mn_v = b'_u + f_v mn,$$

* Congruences, p. 317.

† Brussels Paper, p. 28.

where V', V'', etc., denote derivatives. On eliminating b' between the sixth and ninth conditions, on making use of the fifth condition, and on integrating with respect to v there results

$$n = U_1 V e^f, \tag{6}$$

where U_1 is an arbitrary nonvanishing function of u alone. By means of (6) and the fifth and eighth conditions we find, after integrating with respect to v,

$$a = 2f_{uu} - f_u \frac{U_1'}{U_1} - f_u^2 - U_2,$$

where U_2 is still another arbitrary function of u only.

Let us now regard the functions f, V, U, U_1, U_2 as given functions, and let us consider the possibility of defining the coefficients of a system (D) by the following formulas:

$$\begin{aligned}
m &= \frac{f_{uv}e^{-f}}{U_1 V}, & n &= U_1 V e^f. \\
a &= 2f_{uu} - f_u \frac{U_1'}{U_1} - f_u^2 - U_2, & b &= - UV' - UVf_v, \\
c &= f_u, & d &= UV, \\
a' &= 0, & b' &= \frac{V''}{V} + f_{vv} + \frac{V'}{V} f_v, \\
c' &= 0, & d' &= f_v.
\end{aligned} \tag{7}$$

When the coefficients thus defined are substituted in the integrability conditions, all of them are found to be satisfied identically except the sixth. This condition is found to be equivalent to $I = 0$. When it is written out in terms of the given functions, it becomes

$$\begin{aligned}
&\frac{f_{uuuv}}{f_{uv}} - \frac{3f_u f_{uuv}}{f_{uv}} - 3f_{uu} + 3f_u^2 - \frac{2U_1' f_{uuv}}{U_1 f_{uv}} \\
&\qquad + 4f_u \frac{U_1'}{U_1} + 2\left(\frac{U_1'}{U_1}\right)^2 - \frac{U_1''}{U_1} + U_2 = 0.
\end{aligned} \tag{8}$$

Therefore if the functions f, U_1, U_2 satisfy equation (8), formulas (7) furnish the coefficients of a system (D) which satisfies all of the requirements.

Our results may be summarized in the following fundamental theorem. *Let U, U_1, and U_2 be arbitrary functions of u alone such that $UU_1 \neq 0$. Furthermore let f be a function of u and v statisfying equation* (8) *and such that $f_{uv} \neq 0$. Finally let V be an arbitrary nonvanishing function of v alone. Then formulas* (7) *furnish the coefficients of the most general system of the form*

(D) *for which S_y is nondegenerate and nondevelopable and S_z nondegenerate, and for which the axis curves on S_y, defined with reference to the parametric conjugate system, are indeterminate.* The integrability conditions for such a system (D) are all satisfied identically.

III. A SIMPLIFYING TRANSFORMATION.

All transformations of the group (T) are without geometrical significance; we shall therefore employ this group to simply our analysis. The effect of (T) on (D) is to transform (D) into another system of the same form, of which the coefficients have the following values.*

$$
(9)\qquad
\begin{aligned}
\bar{m} &= \frac{\mu}{\lambda}\frac{1}{\beta_v}m, & \bar{n} &= \frac{\lambda}{\mu}\frac{1}{\alpha_u}n,\\
\bar{a} &= \frac{1}{\alpha_u^2}\left(a+\frac{\lambda_u}{\lambda}c-\frac{\lambda_{uu}}{\lambda}\right), & \bar{b}' &= \frac{1}{\beta_v^2}\left(b'+\frac{\mu_v}{\mu}d'-\frac{\mu_{vv}}{\mu}\right),\\
\bar{b} &= \frac{1}{\alpha_u^2}\frac{\mu}{\lambda}\left(b+\frac{\mu_v}{\mu}d\right), & \bar{a}' &= \frac{1}{\beta_v^2}\frac{\lambda}{\mu}\left(a'+\frac{\lambda_u}{\lambda}c'\right),\\
\bar{c} &= \frac{1}{\alpha_u}\left(c-2\frac{\lambda_u}{\lambda}-\frac{\alpha_{uu}}{\alpha_u}\right), & \bar{d}' &= \frac{1}{\beta_v}\left(d'-2\frac{\mu_v}{\mu}-\frac{\beta_{vv}}{\beta_v}\right),\\
\bar{d} &= \frac{\beta_v}{\alpha_u^2}\frac{\mu}{\lambda}d, & \bar{c}' &= \frac{\alpha_u}{\beta_v^2}\frac{\lambda}{\mu}c'.
\end{aligned}
$$

In the first place we observe that the product UV is reduced to unity by a transformation of the group (T) for which

$$\lambda = U, \qquad \mu V = \alpha_u = \beta_v = 1.$$

When this transformation has been made the coefficient $\bar{d}$ of the transformed system beocmes equal to unity, and the integrability conditions reduce to

$$
\begin{gathered}
c = f_u, \qquad d' = f_v, \qquad b = -f_v, \qquad a' = 0,\\
mn = f_{uv}, \qquad b' = f_{vv}, \qquad n_{vv} = f_v n_v + nb',\\
2m_u n + mn_u = a_v + mnf_u,\\
m_v n + 2mn_v = b'_u + mnf_v.
\end{gathered}
$$

On eliminating b' between the sixth and ninth conditions, on making use of the fifth condition, and on integrating with respect to v, there results

$$n = U_1 e^f.$$

The largest subgroup of the group (T) which preserves the condition $d = 1$ is given by the relations

$$\alpha_u^2\lambda = \beta_v\mu = \text{const.},$$

* Brussels Paper, Equations (16) and (22).

where λ and μ are still arbitrary functions. The effect of this subgroup on the coefficient n is given by

$$\bar{n} = \frac{\beta_v}{\alpha_u^3} n = \frac{\beta_v}{\alpha_u^3} U_1 e^f.$$

Therefore the function U_1 is reduced to unity by a transformation for which

$$\alpha_u^2 \lambda = \mu = 1, \qquad \alpha_u^3 = U_1, \qquad \beta_v = 1.$$

When this transformation has been made, the fifth integrability condition gives

$$m = f_{uv} e^{-f},$$

and then the relation $I = 0$ becomes

$$a = \frac{f_{uuuv}}{f_{uv}} - \frac{3 f_v f_{uuv}}{f_{uv}} - f_{uu} + 2 f_u^2. \tag{10}$$

Now regarding f as a given function, we consider the possibility of defining the coefficients of a system (D) by the following formulas:

$$\begin{aligned}
m &= f_{uv} e^{-f}, & n &= e^f, \\
a &= \frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - f_{uu} + 2 f_u^2, & b &= -f_v, \\
c &= f_u, & d &= 1, \\
a' &= 0, & b' &= f_{vv}, \\
c' &= 0, & d' &= f_v.
\end{aligned} \tag{11}$$

When the coefficients thus defined are substituted in the integrability conditions, all of them are found to be satisfied identically except the eighth, which becomes

$$\frac{f_{uuuvv}}{f_{uv}} - \frac{3 f_u f_{uuvv}}{f_{uv}} - \frac{f_{uuuv} f_{uvv}}{(f_{uv})^2} + \frac{3 f_u f_{uuv} f_{uvv}}{(f_{uv})^2} - 6 f_{uuv} + 6 f_u f_{uv} = 0. \tag{12}$$

Therefore if f satisfies equation (12), then formulas (11) furnish the coefficients of a system (D) such that the axis curves of the parametric conjugate system are indeterminate.

The apparently formidable equation (12) can be very much simplified. On rearranging the terms and integrating with respect to v, we obtain

$$\frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - 3 f_{uu} + 3 f_u^2 + U_2 = 0, \tag{13}$$

where U_2 is an arbitrary function of u alone. If equation (13) be multiplied through by f_{uv}, it can be integrated with respect to v, giving

$$f_{uuu} - 3 f_u f_{uu} + f_u^3 + U_2 f_u = U_3, \tag{14}$$

where U_3 is another arbitrary function of u alone. Finally we introduce the function F by the definition

$$f = -\log F, \tag{15}$$

whereupon (14) reduces to the linear homogeneous equation

$$\frac{\partial^3 F}{\partial u^3} + U_2 \frac{\partial F}{\partial u} + U_3 F = 0. \tag{16}$$

We remark that as a consequence of (15) we have

$$F^2 f_{uv} + F F_{uv} - F_u F_v = 0.$$

We may now state the following theorem. *Let U_2 and U_3 be arbitrary functions of the single variable u. Let F be a function of u and v which is a solution of* (16) *but which is not a solution of $FF_{uv} - F_u F_v = 0$. Let the function f be determined from $f = -\log F$. Then the most general system* (*D*) *which is of interest in the theory of indeterminate axis curves can be reduced, by means of transformations of group* (*T*), *to the system:*

$$(\Delta) \qquad \begin{aligned} y_v &= f_{uv} e^{-f} z, \qquad z_u = e^f y, \\ y_{uu} &= \left(\frac{f_{uuuv}}{f_{uv}} - \frac{3 f_u f_{uuv}}{f_{uv}} - f_{uu} + 2 f_u^2 \right) y - f_v z + f_u y_u + z_v, \\ z_{vv} &= f_{vv} z + f_v z_v. \end{aligned}$$

The integrability conditions for system (Δ) are all satisfied identically. We shall base our further theory on this system.

IV. The First Laplace Transformed Congruence.

In the theory of indeterminate axis curves, the first Laplace transformed congruence, whose focal surfaces are S_ρ and S_y, and the original congruence, whose focal surfaces are S_y and S_z and whose equations are system (D), are equally important. We shall need to set up the system (D_1) of equations of the first Laplace transformed congruence. But before doing so let us define a new function g by the equation

$$g = 2f - \log f_{uv}. \tag{17}$$

As thus defined, g plays the same rôle in the theory of the first Laplace transformed congruence that f plays in the theory of the original congruence.

By means of equation (17) it is possible to write (13) in the symmetrical form

$$f_u^2 + g_u^2 - f_{uu} - g_{uu} - f_u g_u + U_2 = 0. \tag{18}$$

Differentiation of (18) with respect to v yields

$$2 f_u f_{uv} + 2 g_u g_{uv} - f_{uuv} - g_{uuv} - f_{uv} g_u - f_u g_{uv} = 0. \tag{19}$$

Now the sum of the first, third, and fifth terms of this equation is easily shown to be zero, by using equation (17). Therefore, if we suppose $g_{uv} \neq 0$, an assumption whose geometrical meaning will be made clear later, we have

$$f_u = 2g_u - \frac{g_{uuv}}{g_{uv}}. \tag{20}$$

We obtain, on integrating with respect to u,

$$f = 2g - \log g_{uv} + \log V, \tag{21}$$

where V is an arbitrary function of v alone.

We may make a transformation to reduce V to unity. It is readily verified that all transformations of the group (T) for which

$$\lambda = \alpha_u, \qquad \alpha_u^2\lambda = \beta_v\mu = 1, \tag{22}$$

leave the form of system (Δ) unchanged, although f is thereby transformed according to the formula

$$\bar{f} = f - \log \mu. \tag{23}$$

Moreover, g is transformed according to the formula

$$\bar{g} = g - 3 \log \mu, \tag{24}$$

as is seen upon using equation (17). If then we choose

$$\lambda = 1, \qquad \mu = V^{-\frac{1}{3}}, \qquad \alpha_u = 1, \qquad \beta_v = V^{\frac{1}{3}}, \tag{25}$$

the conditions (22) are satisfied and we find, on making the necessary calculations, that

$$\bar{f} = 2\bar{g} - \log \bar{g}_{\bar{u}\bar{v}},$$

so that the transformed $\bar{V}$ is unity. Equation (17) may now be solved for f in the form

$$f = 2g - \log g_{uv}. \tag{26}$$

When f is eliminated from equation (14), the result is

$$g_{uuu} - 3g_u g_{uu} + g_u^3 + U_2 g_u - U_2' + U_3 = 0. \tag{27}$$

Let us introduce the function G by the definition

$$g = -\log G. \tag{28}$$

Then (27) is equivalent to

$$\frac{\partial^3 G}{\partial u^3} + U_2 \frac{\partial G}{\partial u} + (U_2' - U_3) G = 0. \tag{29}$$

It will be observed that this equation is the Lagrange adjoint of equation (16), which is satisfied by F, the derivatives in both equations being taken with respect to the variable u only.

It is interesting at this point to see that F and G satisfy also certain equations in which differentiations occur only with respect to the variable v. The truth of the equation

$$2f_vf_{uv}+2g_vg_{uv}-f_{vvu}-g_{vvu}-f_{uv}g_v-g_{uv}f_v=0$$

is established by observing that alternate terms taken together form two sums, one of which vanishes in virtue of equation (17), and the other in virtue of (26). Upon integrating with respect to u, we obtain

$$f_v^2+g_v^2-f_{vv}-g_{vv}-f_vg_v+V_2=0,$$

where V_2 is an arbitrary function of v alone. When f is eliminated from this equation, there results

$$\frac{g_{vvvu}}{g_{uv}}-\frac{3g_vg_{vvu}}{g_{uv}}-3g_{vv}+3g_v^2+V_2=0.$$

Upon integrating with respect to u again, we obtain

$$g_{vvv}-3g_vg_{vv}+g_v^3+V_2g_v=V_3, \tag{30}$$

where V_3 is a function of v only. Equation (30) is equivalent to

$$\frac{\partial^3G}{\partial v^3}+V_2\frac{\partial G}{\partial v}+V_3G=0. \tag{31}$$

Similarly, if g is eliminated from (30), there results

$$f_{vvv}-3f_vf_{vv}+f_v^3+V_2f_v-V_2'+V_3=0,$$

and this equation is equivalent to

$$\frac{\partial^3F}{\partial v^3}+V_2\frac{\partial F}{\partial v}+(V_2'-V_3)F=0. \tag{32}$$

Equation (32) is the Lagrange adjoint of (31).

Making use of the function g, let us place*

$$\rho=y_u-(f_u-g_u)y,\qquad \zeta=e^{g-f}y, \tag{33}$$

and find the system (Δ_1) of equations of the first Laplace transformed congruence. On differentiating (33) and employing system (Δ), we find that ρ and ζ satisfy

$$\begin{aligned}&\rho_v=e^g\zeta,\qquad \zeta_u=e^{g-f}\rho,\\ (\Delta_1)\qquad &\rho_{uu}=g_{uu}\rho+g_u\rho_u,\\ &\zeta_{vv}=-g_u\rho+(g_{vv}-f_{vv}-f_vg_v+f_v^2)\zeta+\rho_u+g_v\zeta_v.\end{aligned}$$

* Brussels Paper, pp. 66–67.

V. GEOMETRICAL THEOREMS.

The three covariants ρ, σ, ϵ for system (Δ), when written in terms of f and g, are

$$\rho = y_u - (f_u - g_u)y, \qquad \sigma = z_v - f_v z, \qquad \epsilon = \rho_u - g_u \rho. \tag{34}$$

These equations, together with systems (Δ) and (Δ_1) will now be used to deduce a number of geometric propositions.

From the first of equations (34) we find, by differentiation with respect to u and use of system (Δ), that $\epsilon = \sigma$. Moreover we find $\sigma_u = \sigma_v = 0$. Therefore, *if the axis curves of a conjugate net are indeterminate, the second and minus second Laplace transforms of the net are identical and reduce to a single fixed point.*

The third equation of system (Δ_1) shows that the lines $v =$ const. on S_ρ are straight lines, while the third of equations (34) shows that these lines all pass through P_ϵ. Therefore the surface S_ρ is a cone with its vertex at P_ϵ. In the same way we conclude from the fourth equation of system (Δ) that the lines $u =$ const. on S_z are straight lines, and from the second of equations (34) that these lines all pass through P_σ. Therefore, *if the axis curves of a conjugate net are indeterminate, the first and minus first Laplace transforms of the net are on two cones with a common vertex at the fixed point into which the second and minus second Laplace transforms degenerate.*

Inspection of the covariant (C), namely

$$c'd_1 y^2 - mIy\tau - dm\tau^2, \tag{C bis}$$

shows that, when the axis curves on S_y are indeterminate, the foci of the axis of P_y coincide at the point P_τ. Moreover equation (2) shows that P_τ coincides with the point P_σ. Therefore *if the axis curves of a conjugate system on a surface are indeterminate, the axis congruence reduces to a bundle of lines with its vertex at the fixed point into which the second and minus second Laplace transforms of the surface degenerate.*

Let us differentiate twice with respect to v the first equation of system (Δ). From the equations thus obtained let us eliminate z, z_v, and z_{vv}, using the fourth equation of system (Δ). There results

$$y_{vvv} + (2g_v - 3f_v)y_{vv} + (g_{vv} + g_v^2 + 2f_v^2 - 2f_{vv} - 3f_v g_v)y_v = 0, \tag{35}$$

the equation of the curves $u =$ const. on S_y. Since this equation is of the third order we conclude that the curves $u =$ const. are plane curves. Similarly we differentiate once with respect to u the third equation of system (Δ) and eliminate z, z_v and z_{uv} to obtain

$$\begin{aligned} y_{uuu} - f_u y_{uu} + (g_{uu} - 2f_{uu} + f_u g_u - g_u^2)y_u \\ + (g_{uuu} - f_{uuu} + f_u g_{uu} + g_u f_{uu} - 2g_u g_{uu})y = 0, \end{aligned} \tag{36}$$

the equation of the curves $v =$ const. on S_y. These curves are also plane curves. In this way we see that, *if the axis curves of a conjugate system are indeterminate, this conjugate system consists of two one-parameter families of plane curves.*

VI. INTEGRATION OF SYSTEM (Δ).

It has been shown that, for system (Δ), the two surfaces S_σ and S_ϵ reduce to the same fixed point, which may be denoted indifferently by P_σ or P_ϵ. Since the fundamental tetrahedron of reference is arbitrary, let us choose it so that P_σ may have (0, 0, 0, 1) for its coördinates. Then the second of equations (34) frunishes

$$z_v^{(k)} - f_v z^{(k)} = 0, \qquad (k = 1, 2, 3),$$
$$z_v^{(4)} - f_v z^{(4)} = 1.$$

After integrating with respect to v, we obtain

$$z^{(k)} = \varphi^{(k)} e^f,$$
$$z^{(4)} = \varphi^{(4)} e^f + e^f \int e^{-f} dv,$$

where the four functions φ are functions of u alone, and are as yet arbitrary. By the indicated quadrature, we mean the definite integral from a fixed lower limit to a variable upper limit, u being regarded as fixed.

We now use the second equation of system (Δ) to determine four values of y, and obtain

$$y^{(k)} = \varphi^{(k)} f_u + \varphi_u^{(k)},$$
$$y^{(4)} = \varphi^{(4)} f_u + \varphi_u^{(4)} + f_u \int e^{-f} dv - \int e^{-f} f_u dv.$$

When corresponding values of y and z are substituted in the first and fourth equations of system (Δ), these equations are found to be satisfied identically without restriction on the functions φ. But when corresponding values of y and z are substituted in the third equation, it is found that the functions φ must be solutions of certain ordinary differential equations. In fact $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3)$, *constitute three pairs of linearly independent solutions of system* (Δ) *if, and only if, the* $\varphi^{(k)}$ *are three linearly independent solutions of the equation**

$$\frac{d^3\varphi}{du^3} + U_2 \frac{d\varphi}{du} + U_3 \varphi = 0, \tag{37}$$

* Since the second derivative is missing, the Wronskian of any three linearly independent solutions is a constant, which may be supposed to be unity.

and $(y^{(4)}, z^{(4)})$ *are a pair of solutions of system* (Δ) *if, and only if,* $\varphi^{(4)}$ *is a solution of the equation*

$$\frac{d^3\varphi}{du^3} + U_2\frac{d\varphi}{du} + U_3\varphi = 1. \tag{38}$$

Equations (37) and (16) are to be compared.

Let us denote the three second order Wronskians of the $\varphi^{(k)}$ by $U^{(k)}$, so that, for instance,

$$U^{(1)} = \varphi_u^{(2)}\varphi^{(3)} - \varphi_u^{(3)}\varphi^{(2)}.$$

Then the functions $U^{(k)}$ are solutions of the Lagrange adjoint of (37), namely

$$\frac{d^3U}{du^3} + U_2\frac{dU}{du} + (U_2' - U_3)U = 0. \tag{39}$$

This equation is to be compared with equation (29) which is satisfied by G.

We remark further that comparison of (37) and (38) shows that $\varphi^{(4)}$ can be expressed in terms of the $\varphi^{(k)}$ by the well-known method of *variation of parameters*. We find in this way

$$\varphi^{(4)} = \sum_{k=1}^{3}\left(\int U^{(k)}du + c^{(k)}\right)\varphi^{(k)}, \tag{40}$$

where the $c^{(k)}$ are arbitrary constants, and the indicated quadratures denote again definite integrals from a common fixed lower limit to a variable upper limit.

In the process of integrating system (Δ) we have used the second of equations (34), but we might just as well have used the third. This equation gives

$$\rho_u^{(k)} - g_u\rho^{(k)} = 0, \qquad (k = 1, 2, 3),$$
$$\rho_u^{(4)} - g_u\rho^{(4)} = 1.$$

On integrating with respect to u we obtain

$$\rho^{(k)} = \psi^{(k)}e^g,$$
$$\rho^{(4)} = \psi^{(4)}e^g + e^g\int e^{-g}du,$$

where the functions ψ are functions of v alone, and are as yet arbitrary. The first equation of system (Δ_1) gives

$$\zeta^{(k)} = \psi_v^{(k)} + g_v\psi^{(k)},$$
$$\zeta^{(4)} = \psi_v^{(4)} + g_v\psi^{(4)} + g_v\int e^{-g}du - \int e^{-g}g_v du.$$

Then the second of equations (33) furnishes four values for y, and the first of system (Δ) furnishes four values for z.

We find that the $\psi^{(k)}$ are solutions of

$$\frac{d^3\psi}{dv^3} + V_2\frac{d\psi}{dv} + V_3\psi = 0, \tag{41}$$

and $\psi^{(4)}$ is a solution of

$$\frac{d^3\psi}{dv^3} + V_2\frac{d\psi}{dv} + V_3\psi = 1. \tag{42}$$

We denote the three Wronskians of the $\psi^{(k)}$ by $V^{(k)}$, so that, for instance,

$$V^{(1)} = \psi^{(2)}\psi^{(3)}_v - \psi^{(3)}\psi^{(2)}_v.$$

Then by variation of parameters we find

$$\psi^{(4)} = \sum_{k=1}^{3}\left(\int V^{(k)}dv + c^{(k)}\right)\psi^{(k)}. \tag{43}$$

Equation (41) should be compared with equation (31) which is satisfied by G, while the $V^{(k)}$ satisfy the Lagrange adjoint of (41), namely

$$\frac{d^3V}{dv^3} + V_2\frac{dV}{dv} + (V_2' - V_3)V = 0. \tag{44}$$

This equation is to be compared with (32), which is satisfied by F.

We are able to express F and G in terms of the functions $\varphi^{(k)}$ and $\psi^{(k)}$. In fact, F is a solution of (16) and (32), while G is a solution of (29) and (31). Now, reference to (37), (39), (41), and (44) shows that we have

$$F = \sum_{k=1}^{3} V^{(k)}\varphi^{(k)}, \qquad G = \sum_{k=1}^{3} U^{(k)}\psi^{(k)}, \tag{45}$$

where the $V^{(k)}$ and $U^{(k)}$ are the Wronskians previously defined.

Recalling then that $f = -\log F$, $g = -\log G$ we can show that the values obtained for y by both methods of integrating system (Δ) reduce to the following:

$$\begin{aligned} Fy^{(1)} &= U^{(3)}V^{(2)} - U^{(2)}V^{(3)}, \\ Fy^{(2)} &= U^{(1)}V^{(3)} - U^{(3)}V^{(1)}, \\ Fy^{(3)} &= U^{(2)}V^{(1)} - U^{(1)}V^{(2)}, \\ 1y^{(4)} &= \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}\right)y^{(k)}. \end{aligned} \tag{46}$$

The values for z, obtained by both methods, reduce to

$$Fz^{(k)} = \varphi^{(k)}, \qquad (k = 1, 2, 3),$$
$$1z^{(4)} = \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}\right)z^{(k)},$$

while the values for ρ reduce to

$$G\rho^{(k)} = \psi^{(k)}, \qquad (k = 1, 2, 3),$$
$$1\rho^{(4)} = \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}\right)\rho^{(k)}.$$

VII. GEOMETRICAL APPLICATIONS.

We shall now proceed to apply formulas (46), which were obtained by integrating system (Δ). These formulas give the coördinates of an arbitrary point P_y on the most general surface S_y which has the property that the axis curves of the parametric conjugate net are indeterminate.

Let us first deduce the equation of the plane tangent to S_y at P_y. This plane is determined by the three points y, y_u, and y_v, and when its equation is found in the ordinary way, it turns out to be

$$x_4 = \sum_{k=1}^{3}\left(\int U^{(k)}du + \int V^{(k)}dv + c^{(k)}x_k\right). \tag{47}$$

This equation has the same form as the last of (46). As is well known, this plane osculates at P_ρ the curve $u =$ const. through P_ρ on the cone S_ρ, and also osculates at P_z the curve $v =$ const. through P_z on the cone S_z. The two families of curves, $u =$ const. on $S\rho$, and $v =$ const. on S_z, may then be said to have the same set of osculating planes, namely the set of tangent planes of S_y.

If we differentiate equation (47) with respect to u, we obtain

$$\sum_{k=1}^{3} U^{(k)}\, x_k = 0. \tag{48}$$

This plane cuts the surface S_y in the curve $u =$ const. which passes through P_y, and moreover is tangent to the cone S_z along the corresponding generator $u =$ const. In the same way, we obtain by differentiating equation (47) with respect to v,

$$\sum_{k=1}^{3} V^{(k)}x_k = 0. \tag{49}$$

This plane cuts S_y in the curve $v =$ const. which passes through P_y, and is tangent to the cone S_ρ along a generator $v =$ const.

The functions $U^{(k)}$ are the coördinates of the enveloping plane of the cone S_z, and the $V^{(k)}$ are the coördinates of the enveloping plane of S_ρ. Reference to equations (46) shows that the coördinates of P_y depend only on these six functions and the three arbitrary constants $c^{(k)}$. So then, except for these three constants, the surface S_y is determined when the cones S_z and S_ρ are given.

Let us seek for a geometrical meaning for the constants $c^{(k)}$. To this end, let us consider the following projective transformation:

$$
\begin{aligned}
\omega x_1' &= x_1 \qquad\qquad\qquad\qquad\qquad , \\
\omega x_2' &= \qquad\quad x_2 \qquad\qquad\qquad , \\
\omega x_3' &= \qquad\qquad\qquad x_3 \qquad , \\
\omega x_4' &= a^{(1)}x_1 + a^{(2)}x_2 + a^{(3)}x_3 + x_4.
\end{aligned} \tag{50}
$$

This transformation is completely characterized by the property of leaving invariant the point (0, 0, 0, 1) and every stright line through this point. It therefore leaves P_σ and the cones S_z and S_ρ invariant. But it changes the surface S_y, so that the first three coördinates of P_y remain the same as given by (46), while the fourth becomes

$$y^{(4)} = \sum_{k=1}^{3} \left(\int U^{(k)} du + \int V^{(k)} dv + c^{(k)} + a^{(k)} \right) y^{(k)}.$$

Therefore assigning various values to the constants $c^{(k)}$ merely amounts to making projective transformations of the simple type (50). In other words, the surface S_y is determined by the cones S_z and S_ρ up to a projective transformation of this type.

We may now summarize our results. *Let there be given two cones with a common vertex. Then there exists a three-parameter family of surfaces, each surface having the following properties:*

1. *The tangent planes of the two cones cut the surface in a conjugate system.*
2. *The axis curves of this system are indeterminate.*
3. *The axis of every point on the surface, defined with reference to this system, passes through the common vertex of the given cones. Moreover, the family of surfaces has the property that any surface of it may be obtained from any other surface of it by means of the projective transformation that leaves invariant the common vertex of the two given cones and also every line through this point.*

VIII. A Special Case.

We have assumed throughout this work that $g_{uv}f_{uv} \neq 0$. The fifth integrability condition, when $c' = 0$ and S_y is nondegenerate, shows that

$f_{uv} = 0$ if, and only if, S_z degenerates into a curve. This curve would be a straight line, since S_z is also a developable. By analogy, $g_{uv} = 0$ if, and only if, S_ρ reduces to a straight line. Now it happens here, as it often does happen, that the essential results of the investigation are true independently of certain restricting hypotheses used in deriving them. For example, equations (46) are valid, even if one or both of the two cones do reduce to straight lines.

Let us now specialize the six functions and three constants appearing in (46) so as to obtain a very simple surface which has on it a conjugate system with indeterminate axis curves. In the last of equations (46) we shall take the lower limit of integration to be zero both for u and for v. Then we shall take

$$\begin{array}{lll} U^{(1)} = -1, & U^{(2)} = 0, & U^{(3)} = u, \\ V^{(1)} = 0, & V^{(2)} = -1, & V^{(3)} = v, \\ c^{(1)} = 0, & c^{(2)} = 0, & c^{(3)} = -\tfrac{1}{2}. \end{array}$$

In this way equations (46) reduce to

$$(51) \qquad \begin{gathered} Fy^{(1)} = -u, \qquad Fy^{(2)} = -v, \qquad Fy^{(4)} = -1, \\ Fy^{(4)} = \tfrac{1}{2}(u^2 + v^2 + 1). \end{gathered}$$

Let us introduce nonhomogeneous coördinates by the equations

$$x = y^{(1)}/y^{(4)}, \qquad y = y^{(2)}/y^{(4)}, \qquad z = y^{(3)}/y^{(4)}.$$

Then from (51) we obtain

$$(52) \quad x = -\frac{2u}{u^2 + v^2 + 1}, \qquad y = -\frac{2v}{u^2 + v^2 + 1}, \qquad z = -\frac{2}{u^2 + v^2 + 1}.$$

Eliminating the parameters u and v, we obtain

$$(53) \qquad x^2 + y^2 + (z + 1)^2 = 1.$$

At the same time equations (48) and (49) reduce to

$$x = uz, \qquad y = vz.$$

Therefore, the unit sphere, tangent to the xy-plane at the origin, is a surface on which there exists a conjugate system with indeterminate axis curves. The pencil of planes through the x-axis cuts the sphere in the curves $v =$ const., and the pencil of planes through the y-axis cuts the sphere in the curves $u =$ const. The cones S_z and S_ρ are simply these two coördinate axes. And the axis of every point on the sphere passes through the origin.

The generalization to quadric surfaces is immediate. *At any point P on any quadric surface select any two tangent lines separating harmonically the two generators through P. With each of these tangents as axis, construct a pencil of planes. The planes of these two pencils cut the quadric in two one-parameter families of plane curves which constitute a conjugate system with indeterminate axis curves. The axis of every point on the quadric, defined with reference to this system, passes through P.*

UNIVERSITY OF WISCONSIN,
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BOUNDARY VALUE AND EXPANSION PROBLEMS: ALGEBRAIC BASIS OF THE THEORY.*

By R. D. Carmichael.

1. *Introduction.*—Linear problems in various forms have been central in the development of extensive chapters of mathematical analysis. Many of the profound phenomena of nature are subject to laws whose expression in mathematical form gives rise to fundamental linear problems of several kinds. Logically the simplest and historically the first to be treated in detail of the linear problems of pure mathematics are those having to do with systems of linear algebraic equations. And these hold the place of greatest importance both on account of their simplicity and complete development and on account of their suggestiveness in other linear problems. In fact it is true that a wide and important range of transcendental linear problems emerge from a direct consideration of the natural limiting cases of algebraic systems under the guidance of current problems of transcendental analysis.

The deep-lying connection between algebraic and transcendental problems of linear character has often afforded a useful help in the investigation of the latter, particularly in the case of integral equations. The object of the present investigation (of which the first portion is here presented) is to treat a considerable range of transcendental boundary value and expansion problems in close connection with certain algebraic problems of which they are the limiting cases.

It is necessary first of all to formulate and solve the algebraic problems with special reference to their use as a heuristic guide in discovering truths and their proofs for the limiting cases which are matters of prime interest. From the outset classic theorems for certain transcendental cases throw a light back upon the algebraic questions and are of great value in formulating and in solving the algebraic problems. Thus we have from the outset a valuable interaction between the algebraic problems on the one hand and the transcendental problems on the other. The great intimacy of this interaction becomes increasingly apparent as the investigation proceeds.

In his researches on integral equations, integro-differential equations, functions of lines, and permutable functions, Volterra† has made frequent

* Presented to the American Mathematical Society, April, 1920.

† See his "Le cours sur les équations intégrales et les équations intégro-différentielles," 1913, where an exposition of his work is given with references to his earlier memoirs.

and consistent use of the guide to the transcendental problems afforded by looking upon them as limiting cases of algebraic problems. This seems to be the first instance of the systematic and profound use of the principle of guidance which in a modified or extended form is made to lead the way in our study of boundary value and expansion problems. The same general principle served Fredholm* as a heuristic guide in his fundamental memoir on integral equations; and Hilbert† carried through the limiting processes which served Fredholm merely as a guide and deduced his results as limiting cases of algebraic propositions when the number of variables becomes infinite.

Though the guide to transcendental problems afforded by the properties of algebraic equations has attracted the greatest attention in the field of integral equations and the closely related matters, it was not here that the method was first employed heuristically. It is indeed the unpublished method which Sturm employed in his fundamental researches on differential equations of the second order. The differential equation with boundary conditions may be looked upon as the limiting case of a corresponding difference equation with boundary conditions, the latter being but an abbreviated form of a certain restricted system of algebraic equations. The passage to the limit from the difference equation to the differential equation may be carried through rigorously in the case of several boundary value problems, as was shown by Porter in 1902 (more than two years before Hilbert took a similar step for integral equations).‡

An examination of the previous study of differential equations from the point of view in consideration brings out the fact that a given differential equation with boundary conditions may be realized in an infinite number of ways as a limiting form of algebraic systems. The investigator must therefore exercise care in choosing that algebraic system of which a maximum number of properties persist if he is to obtain the greatest advantage from the interaction of the two problems. The results in § 6 of the present paper were suggested by classic theorems for differential equations; success in arriving at them required first the choice of a particular form of algebraic system to yield the differential equation and the generalization of this form so as to maintain the requisite characteristics. The scaffolding by which the results were discovered has not been reproduced in the section.

Similarly, one may look upon partial differential equations, integro-

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* *Acta Mathematica*, 27 (1903): 365.

† *Göttingen Nachrichten*, 1904, p. 49.

‡ For references and a fuller discussion of matters referred to in this paragraph the reader may consult Bôcher's lecture before the Fifth International Congress of Mathematicians and his "Les méthodes de Sturm."

differential equations, and various types of mixed equations, each as a limiting form of appropriate algebraic systems. Moreover, one may consider sets of algebraic systems and their limiting forms of such sort as to furnish guidance in the study of systems consisting of differential equations, difference equations, integral equations, and so forth, and any one of many combinations of these. The algebraic theory developed in the present paper is of such sort as to furnish guidance in the investigation of boundary value and expansion problems associated with each of these various types of equations and systems, as I shall make clear in later papers recording the results of the present general investigation.

The matter developed in this paper may be outlined as follows: In § 2 are recorded preliminary considerations relative to sets of adjoint algebraic systems involving a finite number of parameters subject to determination so as to render the systems consistent. The conjugate character of the solutions of these systems is treated in § 3 and properties are established similar to those of orthogonality and biorthogonality in the theory of functions. These results are applied in § 4 in deriving the 'expansions' of sets of constants in terms of conjugate sets. A certain graphic representation of solutions is defined and analyzed in § 5 and is employed in § 6 in deriving and extending certain theorems analogous to the Sturmian zero separation theorems for linear differential equations of the second order. In § 7 there is a discussion of the general character of a fundamental limiting operation (which is to be employed frequently in analyzing the transcendental problems which are the goal of the investigation); and in § 8 a second type of limiting case is briefly treated. Variation of parameters and Green's functions for algebraic systems are the topics discussed in § 9. Finally, in § 10, a method is given for the condensation of the algebraic expansions of § 4 into contour integrals, the methods and the results being analogous to those which Birkhoff (see reference in § 10) has developed for a certain class of differential systems.

2. *Adjoint Algebraic Systems.*—Among the natural ways of introducing the so-called adjoint algebraic system in connection with a given algebraic system is one which is intimately related to the usual method of introducing the adjoint differential expression in connection with a given differential expression. We recall that the differential expression $M(v)$ which is adjoint to a given homogeneous linear differential expression $L(u)$ is gotten by seeking a v such that an indefinite integral of $vL(u)$ can be written as a homogeneous linear differential expression in u of order one lower than that of $L(u)$. Following up the analogy and employing a sum as to i in place of the integral used in the case of differential expressions, we are led to seek the conditions on $y_1, y_2, \cdots, y_n$ such that for given α_{ij} the coefficients

β_{ij} exist of such sort that we have

$$\sum_{i=1}^{n} y_i \sum_{j=1}^{n} \alpha_{ij} x_j \equiv \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_{kj} x_j$$

for each k of the set $1, 2, \cdots, n$. We see that we must have

$$\sum_{i=1}^{n} \alpha_{ik} y_i = 0$$

for every k. Thus with the first of the two sets of linear forms

$$\sum_{j=1}^{n} \alpha_{ij} x_j, \qquad \sum_{j=1}^{n} \alpha_{ji} y_j, \qquad i = 1, 2, \cdots, n,$$

we are led to associate the second. If we start with the latter it is clear that we shall be led in the same way to the former. The two systems of forms are said to be adjoint each to the other.

As an analogue of the classic Lagrange identity [involving $vL(u) - uM(v)$] in the theory of adjoint differential expressions, we have the readily verified identity

$$\sum_{i=1}^{n} \left\{ y_i \sum_{j=1}^{n} \alpha_{ij} x_j - x_i \sum_{j=1}^{n} \alpha_{ji} y_j \right\} \equiv 0,$$

which will be found to serve us in a fundamental way as a guide to the method of approach to numerous problems.

Let us now consider the r homogeneous linear algebraic systems in the unknown quantities x,

$$(1) \quad \sum_{j=1}^{n_h} (a_{0hij} + \lambda_1 a_{1hij} + \lambda_2 a_{2hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad i = 1, 2, \cdots, n_h,$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$. Here $\lambda_1, \lambda_2, \cdots, \lambda_r$ are r parameters by the choice of which the simultaneous consistency of the r systems is to be secured. With these systems let us associate the adjoint systems

$$(2) \quad \sum_{j=1}^{n_h} (a_{0hji} + \lambda_1 a_{1hji} + \lambda_2 a_{2hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \quad i = 1, 2, \cdots, n_h.$$

A necessary and sufficient condition that each of the r systems in (1) shall have solutions not identically zero is that each of the equations

$$(3) \qquad |a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}| = 0, \qquad h = 1, 2, \cdots, r,$$

shall be satisfied, where for a given value of h the first number is a determinant of order n_h whose element in ith row and jth column is that which is written out explicitly. Obviously we have the same necessary and

sufficient condition that each of the r systems in (2) shall have solutions not identically zero.

A set of (finite) values of $\lambda_1, \lambda_2, \cdots, \lambda_r$ for which (1) [(2)] has for each h a solution not identically zero will be called *a set of characteristic values for* (1) [(2)]. We have just seen that (1) and (2) have the same sets of characteristic values, since the sets of characteristic values are the (finite) solutions of system (3) considered as a system of equations for determining $\lambda_1, \lambda_2, \cdots, \lambda_r$.

We shall suppose that the coefficients a are of such sort that each of the expanded determinants in (3) has explicitly present at least one of the parameters λ. Then system (3) is a set of algebraic equations r in number and involving r unknown quantities. We assume furthermore that the coefficients a are such that these r relations are independent in such wise that we have only a finite number of sets of characteristic values.

These sets of characteristic values we shall then denote by

$$\lambda_1^{(\rho)}, \lambda_2^{(\rho)}, \cdots, \lambda_r^{(\rho)}$$

for varying values of ρ, the two ordered sets being distinct for two distinct values of ρ. The corresponding solutions of (1) and (2) we shall then denote by

$$x_{hj}^{(\rho)},\ y_{hj}^{(\rho)},\ j = 1, 2, \cdots, n_h,\ h = 1, 2, \cdots, r. \tag{4}$$

3. *Conjugate Character of the Solutions of* (1) *and* (2). With the notation adopted in the preceding section for solutions of (1) and (2) we have relations which may be written in the form

$$\sum_{j=1}^{n_h} (a_{0hij} + \lambda_1^{(\rho)} a_{1hij} + \cdots + \lambda_r^{(\rho)} a_{rhij}) x_{hj}^{(\rho)} = 0, \qquad i = 1, 2, \cdots, n_h, \tag{5}$$

$$\begin{aligned} &\sum_{j=1}^{n_h} (a_{0hji} + \lambda_1^{(\rho)} a_{1hji} + \cdots + \lambda_r^{(\rho)} a_{rhji}) y_{hj}^{(\sigma)} \\ &\qquad + \sum_{j=1}^{n_h} \sum_{s=1}^{r} (\lambda_s^{(\sigma)} - \lambda_s^{(\rho)}) a_{shji} y_{hj}^{(\sigma)} = 0, \qquad i = 1, 2, \cdots, n_h, \end{aligned} \tag{6}$$

systems being formed for each value h of the set $1, 2, \cdots, r$. For fixed h and i let us multiply the former of these equations member by member by $y_{hi}^{(\sigma)}$ and the latter by $-x_{hi}^{(\rho)}$, and let us then add the two resulting equations member by member. Still holding h fixed in the relation so obtained, sum as to i from 1 to n_h. That part of the first member of the resulting equation which comes from the first member of (5) and the first line of the first member of (6) balances to zero and we have a relation which may readily be reduced to the form

$$\sum_{s=1}^{r} (\lambda_s^{(\sigma)} - \lambda_s^{(\rho)}) \sum_{i=1}^{n_h} \sum_{j=1}^{n_h} a_{shji} x_{hi}^{(\rho)} y_{hj}^{(\sigma)} = 0, \qquad h = 1, 2, \tag{7}$$

Here we have r equations involving the r quantities $\lambda_s^{(\sigma)} - \lambda_s^{(\rho)}$, $s = 1, 2, \cdots, r$, these being not simultaneously zero when ρ and σ are different. Hence the determinant of the coefficients of these quantities in (7) must have the value zero. If we employ different summation variables i and j for the different rows in this determinant, the relation may be put in the form

$$\left| \sum_{i_h=1}^{n_h} \sum_{j_h=1}^{n_h} a_{shj_hi_h} x_{hi_h}^{(\rho)} y_{hj_h}^{(\sigma)} \right| = 0, \qquad \rho \neq \sigma,$$

where the element written explicitly is that in the sth column and the hth row. It is easy to see that this relation may be put into the following more convenient form:

$$\sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1j_1\ldots i_rj_r} \prod_{h=1}^{r} x_{hi_h}^{(\rho)} y_{hj_h}^{(\sigma)} = 0, \qquad \rho \neq \sigma, \tag{8}$$

where $D_{i_1j_1\ldots i_rj_r}$ denotes the determinant

$$D_{i_1j_1\cdots i_rj_r} = \begin{vmatrix} a_{11j_1i_1} & a_{21j_1i_1} & \cdots & a_{r1j_1i_1} \\ a_{12j_2i_2} & a_{22j_2i_2} & \cdots & a_{r2j_2i_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1rj_ri_r} & a_{2rj_ri_r} & \cdots & a_{rrj_ri_r} \end{vmatrix}.$$

This relation for varying ρ and σ, expresses the fundamental conjugate character of the solutions of (1) and (2) with respect to each other.

In (3) let the $\lambda_1, \cdots, \lambda_r$ be replaced by the set of characteristic values $\lambda_1^{(\rho)}, \cdots, \lambda_r^{(\rho)}$. For a given h let $\alpha_{hij}^{(\rho)}$ denote the cofactor of the element in the ith row and jth column of the determinant in the resulting first member. Then, if for each h some one of these cofactors is different from zero, we have a relation of the form

$$x_{h1}^{(\rho)} : x_{h2}^{(\rho)} : \cdots : x_{hn_h}^{(\rho)} = \alpha_{hi1}^{(\rho)} : \alpha_{hi2}^{(\rho)} : \cdots : \alpha_{hin_h}^{(\rho)}.$$

Similarly, we have

$$y_{h1}^{(\rho)} : y_{h2}^{(\rho)} : \cdots : y_{hn_h}^{(\rho)} = \alpha_{h1j}^{(\rho)} : \alpha_{h2j}^{(\rho)} : \cdots : \alpha_{hn_hj}^{(\rho)}.$$

Employing ξ_h and η_h as the (necessarily non-zero) factors of proportionality, we have

$$x_{hk}^{(\rho)} = \xi_h \alpha_{hik}^{(\rho)}, \qquad y_{hk}^{(\rho)} = \eta_h \alpha_{hkj}^{(\rho)}.$$

If we denote the first member of (8) by $S(\rho, \sigma)$ we have then

$$\frac{S(\rho, \rho)}{\prod_{h=1}^{r} \xi_h \eta_h} = \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1j_1\ldots i_rj_r} \prod_{h=1}^{r} \alpha_{hii_h}^{(\rho)} \alpha_{hj_hj}^{(\rho)}. \tag{9}$$

Now the second member of (4) depends only on the coefficients in equations (1) and (2). If i and j exist so that this second member is different from zero we have $S(\rho, \rho) \neq 0$.

Since we shall be interested in these algebraic results primarily for their use as a heuristic guide in certain transcendental problems we shall naturally exclude from consideration such exceptional cases as are complicated or otherwise fail to serve our purpose. Consequently we shall suppose now and usually henceforth that *the coefficients of* (1) *and* (2) *are of such character that the second member of* (9) *is different from zero for every* ρ. Then we have the following result:

The first member of (8) *is equal to zero when* $\rho \neq \sigma$ *and is different from zero when* $\rho = \sigma$.

In the special case in which (1) is self-adjoint and the x's and the y's, for a given ρ, are taken to be the same, it is clear that the condition that all determinants in (8) shall have the same sign (excluding therefore zero values) is a sufficient condition for the non-vanishing of the first member of (8) when $\rho = \sigma$.

In order to bring out clearly the nature of the conditions of conjugacy which we have just established let us consider the special case in which $r = 1$. Systems (1) and (2) may then be conveniently written in the form

$$(10) \quad \sum_{j=1}^{n} (c_{ij} + \lambda b_{ij}) x_j = 0, \qquad \sum_{j=1}^{n} (c_{ji} + \lambda b_{ji}) y_j = 0, \qquad i = 1, 2, \cdots, n.$$

The conditions of conjugacy reduce to

$$(11) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} x_i^{(\rho)} y_j^{(\sigma)} \begin{cases} = 0 \text{ if } \sigma \neq \rho, \\ \neq 0 \text{ if } \sigma = \rho. \end{cases}$$

If we have the more special case in which $b_{ii} = b_i$, $b_{ij} = 0$ if $j \neq i$, these conditions become

$$(12) \qquad \sum_{i=1}^{n} b_i x_i^{(\rho)} y_i^{(\sigma)} \begin{cases} = 0 \text{ if } \sigma \neq \rho, \\ \neq 0 \text{ if } \sigma = \rho. \end{cases}$$

When $b_i = 1$ for every i these become merely the usual conditions of bi-orthogonality; and these in turn reduce to the usual conditions of orthogonality in case the two systems in (10) are identical and the same solutions x and y are taken for the two systems. From this it is seen that our formulæ give extensive generalizations of classic elementary relations of wide usefulness.

Let us consider a special case of equations (1) and (2) giving a rather simple generalization of the results in the preceding paragraph. Let us suppose that each system in (1) contains but a single parameter, say that λ_h is the only parameter in the system for a given value of h. Then the r

equations (3) involve each a single one of the r parameters $\lambda_1, \cdots, \lambda_r$, so that the possible values of any given parameter are obtained by solving a single equation explicitly given. All possible combinations of solutions are allowable in forming the various sets of characteristic values. The determinants in (8) reduce in the present case to their diagonal elements, so that we have for this case the simpler conditions of conjugacy

$$(13) \qquad \sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r}\left(\prod_{h=1}^{r} a_{hhj_hi_h}x_{hi_h}^{(\rho)}y_{hj_h}^{(\sigma)}\right)\begin{cases} = 0 \text{ if } \sigma \neq \rho, \\ \neq 0 \text{ if } \sigma = \rho. \end{cases}$$

If for a given subscript k no characteristic value of λ_k is equal to zero we may divide systems (1) and (2) by λ_k and so introduce a new set of r parameters $1/\lambda_k$, λ_i/λ_k for $i \neq k$. These enter linearly and in the same way as the original parameters. It is easy to see that the solutions x and y are the same as before. Consequently we have new sets of relations similar to (8) and (13) and differing from them only in having a_{khij} replaced throughout by a_{0hij}.

4. *Expansions by Use of Conjugate Sets.*—Let m be the number of sets of characteristic values of $\lambda_1, \cdots, \lambda_r$ and let us suppose that the constants $z_{i_1i_2\cdots i_r}$ for i_h varying from 1 to n_h for each h may be "expanded" in the form

$$(14) \qquad z_{i_1i_2\cdots i_r} = \sum_{k=1}^{m} c_k x_{1i_1}^{(k)}x_{2i_2}^{(k)}\cdots x_{ri_r}^{(k)},$$

where the c_k are independent of the subscripts i_h. Then the properties of conjugacy developed in § 3 are available for an immediate determination of the values of the c_k—much after the manner of determining the coefficients of a Fourier expansion. For this purpose we multiply both sides of equation (14) by $y_{1j_1}^{(l)}\cdots y_{rj_r}^{(l)}$ times the determinant $D_{i_1j_1\cdots i_rj_r}$ explicitly written in connection with (8) and in the resulting equation sum as to the i and the j. In view of the conditions of conjugacy the resulting quantity in the second member depends on only one of the constants c_k, namely c_l; and we then have

$$(15) \qquad c_l = \frac{\displaystyle\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r}\left(D_{i_1j_1\cdots i_rj_r}z_{i_1i_2\cdots i_r}\prod_{h=1}^{r}y_{hj_h}^{(l)}\right)}{\displaystyle\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r}\left(D_{i_1j_1\cdots i_rj_r}\prod_{h=1}^{r}x_{hi_h}^{(l)}y_{hj_h}^{(l)}\right)}.$$

It is easy to obtain broad sufficient conditions for the validity of the "expansion" (14) for any given set $z_{i_1i_2\cdots i_r}$ of $n_1n_2\cdots n_r$ constants. We suppose that $m = n_1n_2\cdots n_r$ so that the number of distinct solutions of (3) is equal to the product of the degrees of the several equations in that system. If then we look upon (14) as a system of m equations for determining

junction of n points in our figure, one on each of the axes, the ith one being on the ith axis and at a distance $|u_i|$ from the original segment and above it or below it according as u_i is positive or negative. Now join by straight line segments the point on each interior axis to the points on the two adjacent axes. We thus obtain a broken line. We shall say that this broken line is the graphic representation* of the point $(u_1 u_2 \cdots u_n)$ in n dimensions or of the set of constants $u_1, u_2, \cdots, u_n$. The points at which (values of s for which) this broken line cuts the original line segment (viewed as the axis of s) we shall call the *zeros* of the set u_i, in analogy with the usual terminology for the zeros of a function $u(s)$.

Now let us suppose that the same set of axes is used for the representation of the n sets $x_i^{(k)}$, $k = 1, 2, \cdots, n$, and the set z_i of equation (16). Let $x^{(k)}(s)$, $k = 1, 2, \cdots, n$, and $z(s)$ denote the functions of the continuous variable s represented by the broken lines corresponding as indicated above to the foregoing sets of constants. Then we shall show that

$$z(s) = \sum_{k=1}^{n} c_k x^{(k)}(s), \tag{20}$$

where the constants c_k have the same values as in (16). For this purpose it is sufficient to consider the graph between two consecutive vertical axes, say between the ith and the $(i+1)$th. Let s_i and s_{i+1} be the values of s at the intersections of these vertical axes with the s-axis. Then between the two vertical axes in consideration we have

$$z(s) = z_i + \frac{s - s_i}{s_{i+1} - s_i}(z_{i+1} - z_i), \qquad x^{(k)}(s) = x_i^{(k)} + \frac{s - s_i}{s_{i+1} - s_i}(x_{i+1}^{(k)} - x_i^{(k)}),$$
$$k = 1, 2, \cdots, n.$$

From these relations and from (16) with the subscripts i and $i + 1$ it follows that (20) is valid for the interval in consideration. Hence (20) is valid for all values of s on the range $a \leqq s \leqq b$ of the original line segment.

Let us consider the like matter for functions of two subscripts and expansions of the form (19). For the representation of u_{ij} for $i = 1, 2, \cdots, \mu$, and $j = 1, 2, \cdots, \nu$, we shall start from ν planes, one for each value of j. In each of these we use for s the same range $a \leqq s \leqq b$ and place the planes in order $1, 2, \cdots, \nu$ for j each directly in front of the preceding one and arrange the vertical axes so that the axes in any one plane are the orthogonal projections on that plane of the axes in any other plane. Then for fixed j we extend the u_{ij} to $u_j(s)$ by linear interpolation as before. Then for each value of s we connect the points $u_j(s)$, $j = 1, 2, \cdots, \nu$, by straight

* Essentially the same graphic representation has been employed by M. B. Porter in *Annals of Mathematics* (2), 3 (1901), p. 56.

equations the relations

$$(24)\qquad \Delta_{kl}\begin{vmatrix} u_m & u_l \\ v_m & v_l \end{vmatrix} = \pm \Delta_{ml}\begin{vmatrix} u_k & u_l \\ v_k & v_l \end{vmatrix};$$

and, in particular, the relations

$$\Delta_{k,\,k+1}\begin{vmatrix} u_{k+1} & u_{k+2} \\ v_{k+1} & v_{k+2} \end{vmatrix} = \Delta_{k+1,\,k+2}\begin{vmatrix} u_k & u_{k+1} \\ v_k & v_{k+1} \end{vmatrix}, \qquad k = 1, 2, \cdots, n.$$

Hence, if $\Delta_{i,\,i+1}$ for a given range of consecutive values of the integer is i of one sign (excluding the value zero), then the determinant w_i,

$$w_i = \begin{vmatrix} u_i & u_{i+1} \\ v_i & v_{i+1} \end{vmatrix},$$

is of one sign for every i of the same range. It is obvious that this result also may be appropriately extended to the more general case mentioned in the preceding paragraph.

Let us now prove the following theorem:

Suppose that $\Delta_{i,\,i+1}$ for a given range R of consecutive values of the integer i is of one sign (and hence not zero). Let k and l be two integers of this range, and suppose that u_k and u_{k+1} are not of the same sign and that if either of them is zero it is u_k (in which case u_{k+1} is certainly different from zero). Suppose again that u_l and u_{l+1} are not of the same sign, and that l is the smallest value of the subscript greater than k for which this is so. Then an integer h exists, $k \leqq h \leqq l$, such that v_h and v_{h+1} are not of the same sign. Moreover, there are not two such values of h except possibly for the pair $h = k$, $h = l$.

The last statement follows readily from the preceding part of the paragraph; for, if it were not true, the earlier part of the theorem could be applied with u and v interchanged so as to show that the consecutive character of u_k and u_l is not maintained.

For the given range R of values of i the determinant w_i is of one sign, as we have already seen. Without loss of generality we may suppose that $w_i > 0$, since if it were less than zero we should merely have to change the sign of every u_i or of every v_i. Then we have

$$u_i v_{i+1} > u_{i+1} v_i.$$

Let us first consider the case in which $u_k = 0$. Then v_k is of the opposite sign to u_{k+1}. Without loss of generality in argument we may (and we will) take u_{k+1} positive and v_k negative. Then $u_{k+1}, \cdots, u_l$ are all positive while u_{l+1} is zero or negative. If v_l is zero or negative we see from the inequality $u_l v_{l+1} > u_{l+1} v_l$ that v_{l+1} is positive. Hence either v_l or v_{l+1} is positive. Hence there is a change of sign in the sequence $v_k, v_{k+1}, \cdots, v_{l+1}$ so that the statement in consideration is true for the case when $u_k = 0$.

If $u_k \neq 0$ we may take it to be negative. Then $u_{k+1}, \cdots, u_l$ are positive and u_{l+1} is zero or negative. From the inequality $u_k v_{k+1} > u_{k+1} v_k$ we see that if v_k is positive then v_{k+1} is negative and the truth of the statement in consideration is granted. It remains to consider the case when v_k is negative. Either the statement is verified or we have $v_{k+1}, \cdots, v_l$ all negative. In the latter case we see from the relation $u_l v_{l+1} > u_{l+1} v_l$ that the second member is zero or positive and hence that v_{l+1} is positive. Hence v_l and v_{l+1} are of different signs and the statement is verified in this case.

This completes the proof of the statement in consideration.

Let us now consider the relative distribution of the zeros of the functions $u(s)$ and $v(s)$ gotten from the set of constants u_i and v_i by linear interpolation after the method of § 5. As in the immediately preceding discussion we confine attention to a sequence of consecutive intervals corresponding to a range R of consecutive values of the subscript i for which $\Delta_{i,\ i+1}$ is of one sign. From the results just proved in the foregoing paragraphs it follows that there is a zero of $v(s)$ between two consecutive zeros of $u(s)$ except possibly for the case when the h of these paragraphs has the value k or l.

Let us consider the case when $h = k$. We have

$$u_k v_{k+1} > u_{k+1} v_k,$$

while v_{k+1} and v_k are not of the same sign and are not both zero. If $u_k = 0$ we have a case when $v(s)$ has a zero between the two given consecutive zeros of $u(s)$. Then consider the case when $u_k \neq 0$. Then u_k and u_{k+1} are of different signs; and we may take the former to be negative and the latter positive without loss of generality in argument, and this we do. If in the interval joining the points corresponding to k and $k + 1$ the zero of $v(s)$ is to the right of that of $u(s)$ we have a zero of $v(s)$ between the two consecutive zeros of $u(s)$. We then consider further the other case, namely, that in which the zero of $v(s)$ in the interval in question is to the left of that of $u(s)$. In this case $v_{k+1} \neq 0$. If v_{k+1} is positive we have

$$\frac{u_k}{u_{k+1}} > \frac{v_k}{v_{k+1}},$$

which is contrary to the present hypothesis that the zero of $v(s)$ in the interval in question is to the left of that of $u(s)$. Hence we must now have v_{k+1} negative. Then either $v(s)$ vanishes between the two consecutive zeros of $u(s)$ or v_l is negative. We consider the latter possibility. Now u_l is positive. Hence from the relation $u_l v_{l+1} > u_{l+1} v_l$ we have

$$\frac{v_{l+1}}{v_l} < \frac{u_{l+1}}{u_l}.$$

From this and the fact that u_l and u_{l+1} are not of the same sign it follows readily that the zero of $v(s)$ in the interval from l to $l+1$ is to the left of that of $u(s)$ in the interval, and hence in this case that a zero of $v(s)$ occurs between two consecutive zeros of $u(s)$. We have now exhausted all the possibilities for the case when $h = k$ and have found that in this case a zero of $v(s)$ certainly occurs between two consecutive zeros of $u(s)$.

The case when $h = l$ may be treated in a similar way, with the same conclusion.

From this it follows that there is always at least one zero of $v(s)$ between two consecutive zeros of $u(s)$ on the interval in consideration. There can not be more than one; for, if so, the result could be applied with u and v interchanged so as to lead to the conclusion that $u(s)$ should have a zero between two consecutive zeros of the same function.

The result thus obtained may be formulated into the following theorem:

THEOREM. *Let $\Delta_{i,\,i+1}$ for a given range R of consecutive values of the integer i be of one sign and let I denote the interval of the s-axis corresponding to this range of i in the sense of the treatment in* § 5. *Let u_i and v_i be two linearly independent solutions of system* (22); *and let these solutions be extended by the method of linear interpolation employed in* § 5 *to the functions $u(s)$ and $v(s)$. Then on the interval I the zeros of $u(s)$ and $v(s)$ separate each other.*

Let us now consider more generally a system of equations

$$(25)\qquad \sum_{j=1}^{n+h} a_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n,$$

in the $n+h$ unknown quantities $x_1, x_2, \cdots, x_{n+h}$, where $h \geqq 2$; and let us suppose that the matrix of this system is of rank n. The system then has h linearly independent solutions; we designate such a set by

$$x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(h)}.$$

Let D_i be the determinant of order n of the square matrix gotten from the matrix of coefficients in (25) by striking out h consecutive columns beginning with the ith, let R be a range of consecutive values of the subscript i for which D_i is of one sign (zero values being excluded); and let I denote the interval of the s-axis corresponding to this range of i in the sense of the treatment in § 5.

Now let w_i denote the determinant

$$(26)\qquad w_i = \begin{vmatrix} x_i^{(1)} & x_{i+1}^{(1)} & \cdots & x_{i+h-1}^{(1)} \\ x_i^{(2)} & x_{i+1}^{(2)} & \cdots & x_{i+h-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ x_i^{(h)} & x_{i+1}^{(h)} & \cdots & x_{i+h-1}^{(h)} \end{vmatrix} = \begin{vmatrix} x_i^{(1)} & \Delta x_i^{(1)} & \cdots & \Delta^{h-1}x_i^{(1)} \\ x_i^{(2)} & \Delta x_i^{(2)} & \cdots & \Delta^{h-1}x_i^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ x_i^{(h)} & \Delta x_i^{(h)} & \cdots & \Delta^{h-1}x_i^{(h)} \end{vmatrix},$$

where $\Delta, \Delta^2, \cdots, \Delta^{h-1}$, in the last form of the determinant, denote the first,

second, $\cdots$, $(h-1)$th differences of the x_i with respect to the variable i. It is obvious that the two forms of the determinant have the same value.

Since D_i is of one sign for i on R it follows from the earlier work in this section that w_i is of one sign for i on R. Let us consider the matrix of any m rows of the first [second] determinant form for w_i in (26) and let $A_{imj}[C_{imj}]$, $j = 1, 2, \cdots, j_m$, be all the m-rowed determinants formed from this matrix. Let $B_{imj}[D_{imj}]$, $j = 1, 2, \cdots, j_m$, be the corresponding algebraic complements. By means of the Laplace development for determinants we then have

$$w_i = \sum_{j=1}^{j_m} A_{imj}B_{imj} = \sum_{j=1}^{j_m} C_{imj}D_{imj}. \tag{27}$$

Suppose that k is a value of i on the range R such that B_{imj} $[D_{imj}]$ is of one sign or zero for each j of the set $j = 1, 2, \cdots, j_m$; and suppose that l is a value of i also on R for which these quantites are of the opposite sign or zero. Then, since w_i is of one sign for i on R, it can not be true that all of the quantities A_{kmj}, A_{lmj} $[C_{kmj}, C_{lmj}]$ are of one sign for the fixed value of m and for j varying over its whole range or indeed over just that part of its range for which the corresponding B_{kmj} or B_{lmj} $[D_{kmj}$ or $D_{lmj}]$ has a value different from zero.

In particular, if B_{kmj} and B_{lmj} $[D_{kmj}$ and $D_{lmj}]$ both have the value zero except for a single value ρ of j and for this value have different signs, then $A_{km\rho}$ and $A_{lm\rho}$ $[C_{km\rho}$ and $C_{lm\rho}]$ have opposite signs. Hence the function $A_{sm\rho}$ $[C_{sm\rho}]$ obtained from $A_{im\rho}$ $[C_{im\rho}]$ by linear interpolation as in § 5 has a zero on that part of the s-axis which corresponds to the range of i from k to l inclusive.

Let us now consider the first determinant in (26) for the case when m is unity and the one row singled out is the last one. Then the A_{i1j}, for varying j, are the elements of the last row of the determinant and the B_{i1j} are their cofactors. If for $i = k$ these cofactors are of one sign or zero and for $i = l$ are of the opposite sign or zero, where k and l are both on the range R of i, then not all of the quantities $x_k^{(h)}, x_{k+1}^{(h)}, \cdots, x_{l+h-1}^{(h)}$ are of one sign. Hence if we form the function $x^{(h)}(s)$ by linear interpolation from $x_i^{(h)}$ this function $x^{(h)}(s)$ has at least one zero in the interval corresponding to the range of i from k to $l+h-1$. Thus for the cases when our results apply we are able by means of $h-1$ linearly independent solutions of (25) to define an interval for i and a corresponding interval on the s-axis such that every solution $x_i^{(h)}$ of (25) which is linearly independent of the first $h-1$ solutions gives rise by linear interpolation to a function $x^{(h)}(s)$ which vanishes on the interval of the s-axis in question. It will be observed that this result is a direct generalization of a portion of that obtained above for the case when $h = 2$.

Let us consider the second determinant in (26) for the case when m is unity and the one row singled out is the last one. Then the C_{i1j}, for varying j, are the elements of the last row and the D_{i1j} are their cofactors. If the latter for $i = k$ are of one sign or zero and for $i = l$ are of the opposite sign or zero, where k and l are values of i on the range R, then the quantities $x_i^{(h)}, \Delta x_i^{(h)}, \cdots, \Delta^{h-1}x_i^{(h)}$, for the two values k and l of i, can not all be of the same sign. In particular, if they are all positive for $i = k$, then one at least of them must fail to be positive at $i = l$; so that at least one of them changes sign or becomes zero in the interval in question.

We may look upon the modification of the first determinant in (26) to produce the second as a single instance of a great variety of transformations of that first determinant, each being made by replacing columns in that determinant by various linear homogeneous combinations of columns (the coefficients being constants or functions of i) in such wise that the non-vanishing of w_i on a range R implies the non-vanishing of the modified determinant $\overline{w}_i$ on the same range R. For every such $\overline{w}_i$ we may obtain a sequence of theorems of the sort just brought to light in the treatment of the two forms given in (26). Such properties as are thus indicated will be useful to us later in the study of certain transcendental problems which are limiting cases of the algebraic problems now in consideration.

Solutions of (25), for a given value of $h > 2$, may also be studied by means of similar systems for a smaller value of h, this smaller value of h being not less than 2. For this purpose we adjoin to (25) one or more additional equations in the same unknown quantities, these equations being linearly independent of each other and of those in (25). In particular, the system may thus be replaced by a new system of the same form but with only two linearly independent solutions. The foregoing results for such a system, when applied to this, yield properties of those solutions of (25) which satisfy the auxiliary conditions, the latter being somewhat analogous to boundary conditions for differential equations. A considerable hold can thus be gotten on a class of solutions of (25). This obvious remark will serve a useful heuristic purpose in studying certain properties of differential equations with boundary conditions.

In this section we have confined ourselves principally to that part of the range of i for which D_i is of one sign. We propose the problem of investigating the matter for the whole range of i from 1 to n and thus of associating the zeros of D_i and of certain combinations of the solutions of (25) for the whole range of i from 1 to n.

7. *General Character of a Fundamental Limiting Operation.*—It is well known, as we have already pointed out more fully, that several important types of functional equations, such as linear differential equations, linear

integral equations, etc., can be exhibited as limiting cases of algebraic systems. In a large range of limiting cases the conjugate character of the solutions of (1) and (2) which we have discussed in § 3 and the expansion by use of conjugate sets treated in § 4 are carried over to the limiting problems in such a way as to afford a definite suggestion of properties to be expected and of methods of analysis for examining them. We shall now consider the general character of the limiting operation in question with reference to the properties just indicated.

The graphic representations employed in § 5 afford a starting-point. Let us take a fixed interval $a \leqq s \leqq b$ of the real s-axis and let us associate with certain points of this interval the discrete values of i and j in equations (1) and (2). For the hth equation in either system we take on (ab) a set of points n_h in number including the points a and b; and we interpolate x_{hj} into a function $x_h(s)$ by linear interpolation as in § 5. Similarly, a function a_{khij} is interpolated into a function $a_{kh}(s, t)$ by a method also described in § 5. Equations (1) and (2) may now be looked upon as establishing relations among the functional values of these functions of s and t at certain points only of the axes of s and t; and so of defining the solution functions at these points and these alone, their definitions being completed by the method of interpolation agreed upon.

The limiting processes described consist essentially in this: let the various numbers n_h (or at least a part of them) increase indefinitely; and as they increase let the functions $x_h(s)$ pass through a corresponding sequence of changes. Various limiting processes are thus set up in the original equations and in their solutions. If these lead to replacing the original equations by well-defined equations and their solutions by well-defined functions, we have in the process a suggestion of a heuristic guide to probable solutions of the limiting problems and to certain probable fundamental properties of these. As we proceed it will be seen that this guide has great usefulness in a considerable range of problems.

We shall usually require that the distribution of basic points on the interval (ab) of the s-axis shall undergo change in such wise that the maximum distance between two consecutive points shall approach zero as n_h becomes infinite. This suggests that the sum as to j in (1) and (2) may sometimes be replaced by an integral as to t from a to b; and, when this is to be done, it is desirable that the coefficients a_{khij} shall be replaced each by a product one factor of which is to correspond to the differential in the limiting case. [If the basis points on the s-axis (or t-axis) are equally spaced one may introduce this needed factor simply by multiplying the equation through by Δs (or Δt), the distance between two consecutive basic points. This is perhaps the easiest way to make the modification in ques-

tion.] In fact such a form has frequently been employed in setting up a limiting process to guide one to the theory of integral equations, first by Volterra and later by others. In numerous ways (of which that just mentioned is one of the most important) one may subject the coefficients in (1) and (2) to subsidiary conditions such that the limiting process in consideration shall lead to desired forms of functional equations as limiting cases, whether differential equations or integral equations or equations of other type.

The several systems in a single set of systems (1) or (2) are independent of each other in such wise that the limiting processes may be applied to each with a certain measure of independence of the others, so that we might have for instance an ordinary differential equation and an integral equation to make up the limiting system of two equations for the case of two parameters λ. Numerous other combinations are logically possible; and several of them are interesting for the results to which they give rise. In a second paper we shall give a precise characterization of several problems to which we are led in carrying out the limiting operation in a variety of ways or combinations of ways.

It remains here to note certain characteristic results for the finite case which persist (in modified form of course) in a great variety of limiting cases.

The property of conjugacy expressed by the equations of system (8) is an important one which persists after certain of the limiting processes just indicated have been carried out. It takes a large number of forms owing to the great variety of ways in which the limiting process may be set up. One of the simplest and at the same time most useful of these is that in which the determinant $D_{i_1 j_1 \dots i_r j_r}$ in (8) becomes a function $D(s_1, t_1, \cdots, s_r, t_r)$ and the summation in (8) is replaced by an integration, the algebraic system having first been taken in a form which is convenient for passing to the limit in a manner to realize such a result. The property of conjugacy may then be expressed by the relations

$$(28)\quad \int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \cdot \prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\sigma)}(t_h) \cdot ds_1 dt_1 \cdots ds_r dt_r = 0, \qquad \rho \neq \sigma.$$

For a wide range of cases we have in the limiting problem an infinite number of sets of characteristic values for the parameters λ and hence an infinite number of sets of functions $x_h^{(\rho)}(s)$, $y_h^{(\rho)}(t)$, $h = 1, \cdots, r$, for varying ρ, for which relation (28) is valid.

One of the most frequently occurring and important special cases of (28) is that which corresponds to equation (12), namely,

$$\int_a^b b(s) x^{(\rho)}(s) y^{(\sigma)}(s) ds = 0, \qquad \rho \neq \sigma.$$

When we proceed in the second paper to certain important instances of the theory here sketched roughly in a general sort of way we shall find that equations essentially like (28) or special cases of it appear in a great variety of forms; but a little examination of each system will show that its equations arise in a natural way as limiting cases of the equations in (8) and that one is led to them by the guide afforded by the general considerations barely outlined here.

Now if the repeated integral in the first member of (28) has a value different from zero when $\rho = \sigma$ (a condition which is realized in a wide range of important cases, as we shall see) and if a function $f(s_1, s_2, \cdots, s_r)$ has an expansion of the form

$$f(s_1, s_2, \cdots, s_r) = \sum_{\rho=1}^{\infty} c_\rho x_1^{(\rho)}(s_1) x_2^{(\rho)}(s_2) \cdots x_r^{(\rho)}(s_r),$$

where the c_ρ are constants and the values of ρ correspond to the various sets of characteristic values of the parameters λ, then the coefficients c_ρ are readily found by a method analogous to that by which the Fourier coefficients for a function are found; and we have in fact

$$c_\rho = \frac{\displaystyle\int_a^b \int_a^b \cdots \int_a^b f(s_1, s_2, \cdots, s_r) D(s_1, t_1, \cdots, s_r, t_r) y_1^{(\rho)}(t_1) y_2^{(\rho)}(t_2) \cdots y_r^{(\rho)}(t_r) ds_1 dt_1 \cdots ds_r dt_r}{\displaystyle\int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \cdot \prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\rho)}(t_h) \cdot ds_1 dt_1 \cdots ds_r dt_r}.$$

In each instance of our theory this formal expansion or the requisite modification of it is used to set the basic expansion problem associated with the instance in consideration.

In the foregoing discussion the number of variables in the function f to be expanded is equal to the number of parameters in the algebraic problem from which we passed to the transcendental problem. This correspondence is not essential to the nature of the process involved. The desired extension can best be brought out by starting from the particular expansion (19). We may carry out our method of linear interpolation as at the close of § 5 and obtain the expansion (21). If now we proceed to the limiting case in such wise that μ and ν simultaneously approach infinity and if we require that the distribution of axis points on the s-axis and the t-axis shall have the usual property of condensation and if the number of distinct characteristic values approaches infinity during the process, we shall be led heuristically to an expansion of the form

$$f(s, t) = \sum_{k=1}^{\infty} c_k x^{(k)}(s, t),$$

where the x's are now solutions of the limiting problem. It is easy to see that the properties of conjugacy are maintained (under appropriate circumstances) and that formulæ are therefore easily found for the values of the c_k. We do not attempt to investigate in general the range of validity of these results. We shall later find in them a useful heuristic guide to problems in partial differential equations; and for the cases involved in such applications we shall analyze the matter of range of validity.

In a similar way we may obtain numerous modifications of the foregoing expansion of $f(s_1, s_2, \cdots, s_h)$ by starting from the modifications of equations (14) suggested by the method of the preceding paragraph for the simplest special case and proceeding in a similar manner.

Another method of proceeding from (19) to a limiting case will be very useful to us in dealing with a system of ordinary linear differential equations. In this method we suppose that the passage to the limit is effected in such way that μ remains fixed so that i is always a discrete variable ranging over the finite set $1, 2, \cdots, \mu$, while ν becomes infinite and the variable j passes in our usual way to the continuous variable s in the limiting equation. Then expansion (19) passes over into the form

$$f_i(s) = \sum_{k=1}^{\infty} c_k x_i^{(k)}(s), \qquad i = 1, 2, \cdots, \mu,$$

a form which may be looked upon as affording the simultaneous expansion of μ given functions $f_i(s)$ of a continuous variable s in terms of μ sets of functions $x_i(s)$, the coefficients c_k being the same in each of the two expansions.* Obviously, numerous more general forms of like expansions of systems of functions of one or more variables may be obtained by passing from the general relation (14) to limiting cases in the various ways sufficiently suggested in the foregoing instances.

The details of the matters suggested in the three preceding paragraphs will be developed only in connection with the instances of certain transcendental problems to which they afford the requisite guide.

It is doubtless true that many other properties of the algebraic systems can be carried over to certain of the important limiting cases. In fact certain of the zero separation theorems of § 6 can be utilized to suggest like theorems for the solutions of certain transcendental problems. Moreover, the interaction of the two types of problems may well be made to work in both directions. A part of the results in § 6 were indeed suggested by classic properties of the solutions of linear differential equations of the

* A special case of expansions of this sort first came to my attention in the dissertation of Dr. C. C. Camp (a manuscript copy of which he was kind enough to lend me); and the suggestion afforded by it has been very useful to me in the present investigation.

second order. Once in hand for the simpler cases these results were easily extended to other cases of algebraic systems and then afforded a new guide to certain related transcendental problems. There is thus a repeated mutual reaction between the two types of problems which is useful in the development of the theory of each.

8. *A Second Type of Limiting Case.*—Let us now consider the r homogeneous linear systems in the infinite number of unknown quantities x,

$$(29)\qquad \sum_{j=1}^{\infty} (a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad i = 1, 2, 3, \cdots,$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$. With these systems let us associate the r adjoint systems

$$(30)\qquad \sum_{j=1}^{\infty} (a_{0hji} + \lambda_1 a_{1hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \qquad i = 1, 2, \cdots.$$

Here $\lambda_1, \lambda_2, \cdots, \lambda_r$ are r parameters by the choice of which the simultaneous consistency of the r systems in either set is to be secured. For these infinite systems a great variety of possibilities may be realized by special methods of defining the coefficients a; and a comprehensive analysis of all mutual possibilities would require the examination of a great many cases. Consequently we shall confine our attention to a range of cases in which the theory may be set forth in the simplest manner.

A set of values of $\lambda_1, \cdots, \lambda_r$ for which (29) [(30)] has a solution not identically zero will be called *a set of characteristic values for* (29) [(30)]. In the case of the finite problem of § 2 we saw that the sets of characteristic values for the two adjoint problems were identical. For the infinite case of this section we shall consider only those systems for which this property is maintained. It is clearly maintained in cases in which those characteristic values can be determined by determinantal equations analogous to equations (3) of § 2. If this method of determining the sets of characteristic values is not suitable we may proceed as follows for a certain range of cases of equations (29) and (30); omit from each system r equations, one for each value of h; solve the remaining equations as non-homogeneous equations, after having assigned a value to one of the unknown quantities (an operation which can be carried out by known processes in a wide range of cases), and suppose that this solution is unique (as indeed it is in a wide range of known cases); now substitute into the equation omitted from each system the values of the unknown quantities so obtained; thus we have for system (29), and also for system (30), r equations involving the r parameters; their solutions give the sets of characteristic values.

Now, we shall suppose that the sets of characteristic values, however

determined (whether by one of the methods indicated or by another), is denumerably infinite and we shall denote them and the corresponding solutions by

$$\lambda_k^{(\rho)}, \quad x_{ki}^{(\rho)}, \quad y_{ki}^{(\rho)}, \qquad \rho = 1, 2, 3, \cdots.$$

Several of the properties developed for the corresponding algebraic problem are now readily carried over to these solutions of a limiting problem, at least in a formal way. Since our present interest in the general procedure arises from intended applications of it to relatively simple instances we shall not undertake an analysis of the range of validity of our results, but shall content ourselves with little more than an exhibition of the formal processes.

If we proceed by a method in all respects similar to that employed in § 3 and if we have before us a case in which the order of infinite summations may be interchanged in ways now needful in carrying out the same method, we come through to certain equations analogous to (8) and expressing the fundamental conjugate character of the solutions of (29) and (30), namely, the equations

$$\text{(31)} \qquad \sum_{i_1=1}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{i_r=1}^{\infty}\sum_{j_r=1}^{\infty} D_{i_1j_1\cdots i_rj_r}\prod_{h=1}^{r} x_{hi_h}^{(\rho)} y_{hj_h}^{(\sigma)} = 0, \qquad \rho \neq \sigma,$$

where $D_{i_1j_1\cdots i_rj_r}$ is notationally the same determinant as that which is represented by this symbol in equation (8).

Following still the method of § 3 it would be possible to name conditions on the coefficients a which are sufficient to insure that the first member of (31) shall have a value different from zero when $\rho = \sigma$. Since we have no need for the explicit form of these conditions we shall not take the space to derive them; we shall merely assume that we have before us a case in which the first member of (31) for $\rho = \sigma$ is always different from zero.

It is obvious that we have particularly simple forms of the conjugacy conditions (31) for cases corresponding to those involved in equations (11), (12), (13); and that we may have modifications of all of the conjugacy conditions analogous to those indicated briefly for the finite case in the paragraph following equation (13).

We are now led to consider the problem of the expansion of a function $z_{i_1i_2\cdots i_r}$ of r discrete arguments, each on the range $1, 2, 3, \cdots$, in terms of the solutions of one set of systems, the expansion being of the form

$$\text{(32)} \qquad z_{i_1i_2\cdots i_r} = \sum_{k=1}^{\infty} c_k x_{1i_1}^{(k)} x_{2i_2}^{(k)} \cdots x_{ri_r}^{(k)},$$

where the coefficients c_k are independent of the variables $i_1, \cdots, i_r$. If the function z has an expansion of this form and if certain changes of order of

summation involved in the obvious process of reckoning are legitimate, we obtain readily for the coefficients c_k the values

$$(33)\quad c_k = \frac{\sum_{i_1=1}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{j_r=1}^{\infty}\sum_{j_r=1}^{\infty} D_{i_1 j_1 \cdots i_r j_r} z_{i_1 i_2 \cdots i_r} \prod_{h=1}^{r} y_{h j_h}^{(k)}}{\sum_{i_1=1}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{j_r=1}^{\infty}\sum_{j_r=1}^{\infty} D_{i_1 j_1 \cdots i_r j_r} \prod_{h=1}^{r} x_{h i_h}^{(k)} y_{h j_h}^{(k)}}, \qquad k = 1, 2, 3, \cdots.$$

The discussion associated with equations (17) and (18) affords suggestions for the formation of special cases or for modifying the form of expansion in (32) and (33).

Even in this transcendental case one may usefully employ a graphic representation of the sort described in § 5. One probably takes most naturally for an s-axis the real axis from 1 to infinity and erects perpendiculars at the points 1, 2, 3, $\cdots$ on this axis, these perpendiculars to serve as the axes of coördinates. One may then fill in the function by linear interpolation as in § 5 and so make a function of a continuous variable. One might conceivably carry out now a limiting process by which the number of perpendiculars in any finite stretch increases indefinitely and the greatest distance between two which are consecutive decreases to zero. One has a limiting case which may be employed to afford suggestions for dealing with differential equations of infinite order. Various modifications and extensions of this graphic representation are suggested by the foregoing treatment for the finite case, particularly that in § 7; but we shall not now pursue the matter further.

Besides the generalization of the problem associated with (1) and (2) to that associated with (29) and (30) one might treat also that in which the number of parameters λ becomes infinite while the number of x's and hence of y's remains finite or that in which the number of λ's and the number of x's and y's simultaneously become infinite; but it appears unlikely that such generalizations will be as useful as that which is indicated in the foregoing paragraphs.

A rather important problem, upon the resolution of which we shall not enter, may be suggested here, namely, the problem of the relative distribution of the zeros of linearly interpolated solutions of infinite systems.

9. *Variation of Parameters and Green's Functions for Algebraic Systems.* —Let us consider a system of n non-homogeneous linear algebraic equations

$$(34)\qquad \sum_{j=1}^{n+h} a_{ij} u_j = b_i, \qquad i = 1, 2, \cdots, n,$$

and the corresponding homogeneous system

$$(35)\qquad \sum_{j=1}^{n+h} a_{ij} x_j = 0, \qquad i = 1, 2, \cdots, n,$$

each in $n+h$ unknown quantities, h being a positive integer, and let us suppose that the rank of the matrix $||a_{ij}||$ is n.

In what follows next we shall assume that the notation is chosen so that the determinant of the square matrix made up of the first n columns of $||a_{ij}||$ has a value different from zero. If the equations in each system are taken in a suitable order (and we shall suppose them to be so written already) it is obviously possible to combine them into a new equivalent system in the same unknown quantities in such way that the new coefficients a_{ij} have the value zero when $j < i$. We shall now suppose further that the original, and hence the new, matrix $||a_{ij}||$ has the property that the determinants of order 1, 2, 3, $\cdots$, n in its lower right-hand corner are all different from zero in value. Then it is possible to make further combinations of equations in either system so as to replace the system by an equivalent system in the same unknown quantities and in such wise that when all this is done the systems (34) and (35) take the forms

$$(36)\qquad \sum_{r=0}^{h} \alpha_{i,\, i+r} u_{i+r} = \beta_i, \qquad \sum_{r=0}^{r} \alpha_{i,\, i+r} x_{i+r} = 0, \qquad i = 1, 2, \cdots, n,$$

where $\alpha_{i,\, i}$ and $\alpha_{i,\, i+h}$ are different from zero for every i. For the equations in this normal form we have a ready method of obtaining the solutions of the non-homogeneous system in terms of a fundamental system of solutions of the homogeneous system analogous to, and indeed abstractly identical with, the classic method of variation of parameters in the theory of differential equations, the latter being in fact a limiting case of the former. This method for the algebraic systems we shall now develop in a brief treatment.

Let us denote by

$$x_j^{(1)}, \quad x_j^{(2)}, \quad \cdots, \quad x_j^{(h)}, \qquad j = 1, 2, \cdots, n+h,$$

a fundamental system of h linearly independent solutions of the homogeneous system in (36), and hence of that in (35). Then we seek a solution of the non-homogeneous system in (36), and hence of that in (34), in the form

$$(37)\qquad \begin{gathered} u_{i+t} = c_i^{(1)} x_{i+t}^{(1)} + c_i^{(2)} x_{i+t}^{(2)} + \cdots + c_i^{(h)} x_{i+t}^{(h)}, \\ t = 0, 1, \cdots, h-1, \qquad i = 1, 2, \cdots, n+1, \end{gathered}$$

where the functions $c_i^{(k)}$ are to be determined. The equations in the system

$$(38)\qquad \begin{gathered} \sum_{k=1}^{h} [c_{i+1}^{(k)} - c_i^{(k)}] x_{i+1+t}^{(k)} = 0, \qquad t = 0, 1, \cdots, h-2, \quad i = 1, 2, \cdots, n, \\ \sum_{k=1}^{h} [c_{i+1}^{(k)} - c_i^{(k)}] x_{i+1}^{(k)} = \frac{\beta_i}{\alpha_{i,\, i+h}}, \qquad i = 1, 2, \cdots, n, \end{gathered}$$

are now obtained as follows: those in the first line come from (37) by comparison of that equation for a given value $i+1$ of i with the equation gotten from (37) when t is replaced by $t+1$; those in the last line are obtained by substituting the value of u from (37) into the non-homogeneous system of (36), the value of u_{i+t} for $t = 0, 1, \cdots, h-1$ being written as in (37) while that of u_{i-h} is taken in the form

$$u_{i+h} = c_i^{(1)} x_{i+h}^{(1)} + \cdots + c_i^{(h)} x_{i+h}^{(h)} + \sum_{k=1}^{h} [c_{i+1}^{(k)} - c_i^{(k)}] x_{i+h}^{(k)}.$$

We look upon equations (38) as furnishing a system for the determination of the functions $c_i^{(k)}$.

The condition that (38), for every value i of the set $1, 2, \cdots, n$ and for every β_i, shall serve for the determination of the quantities in the square brackets is that the determinant Δ_i,

$$\Delta_i = \begin{vmatrix} x_{i+1}^{(1)} & x_{i+1}^{(2)} & \cdots & x_{i+1}^{(h)} \\ x_{i+2}^{(1)} & x_{i+2}^{(2)} & \cdots & x_{i+2}^{(h)} \\ \cdot & \cdot & \cdot & \cdot \\ x_{i+h}^{(1)} & x_{i+h}^{(2)} & \cdots & x_{i+h}^{(h)} \end{vmatrix},$$

shall be different from zero for every i of the set $1, 2, \cdots, n$. But in § 6 we saw that these inequalities are equivalent to the conditions that the determinant of every matrix of order n obtained from $||a_{ij}||$ by striking out h consecutive columns shall be different from zero. These are then necessary conditions for it to be true that every system (34) may be solved by the given method of variation of parameters in terms of a fundamental system of solutions of (35). These conditions, together with those already named in connection with the reduction to the normal systems (36), are also sufficient conditions for it to be true that every system (34) may be solved by the given method, as we shall now show by actually effecting the solution under the named hypotheses.

If we denote by $\Delta_i^{(k)}$ the cofactor of the element in the kth column and last row of Δ_i we have from (38) the relations

$$(39) \qquad c_{i+1}^{(k)} - e_i^{(k)} = \frac{\beta_i}{\alpha_{i,\,i+h}} \frac{\Delta_i^{(h)}}{\Delta_i}, \qquad k = 1, 2, \cdots, h; \qquad i = 1, 2, \cdots, n.$$

If we assign to $c_1^{(k)}$, or to $c_{n+1}^{(k)}$, an arbitrary value $c^{(k)}$, the equations (39) serve by recursion to determine uniquely the $c_i^{(k)}$ in terms of these arbitrary values; and in fact $c_i^{(k)}$ for fixed k and varying i is the arbitrary element $c^{(k)}$ plus a determinate function of i depending only on the b_i, the a_{ij} and the solutions of the homogeneous system. On substituting these values of the c's into (37) we have the general solution u_j of the non-homogeneous system in (36), and hence of system (34).

This solution of (34) may obviously be written in the customary form

$$(40)\quad u_i = c^{(1)}x_i^{(1)} + c^{(2)}x_i^{(2)} + \cdots + c^{(h)}x_i^{(h)} + \bar{u}_i, \qquad i = 1, 2, \cdots, n+h,$$

where $\bar{u}_i$ is a particular solution of (34) and the $c^{(1)}, \cdots, c^{(h)}$ are arbitrary constants, this form of solution being of course always realized under the hypotheses of the first paragraph of the section.

If we take systems of the form (36) where i runs over the infinite set $1, 2, 3, \cdots$ instead of the finite set $1, 2, \cdots, n$, it is easy to see that we have a process in every respect identical with the foregoing except that the recurrence relations which replace (38) are now to be solved by taking the first $c_i^{(k)}$ only (since there is no last), namely $c_1^{(k)}$, as arbitrarily defined. Such a system is intimately connected with the theory of difference equations.

Let us now consider the question of adjoining linear conditions to the systems (34) and (35), these new conditions to be thought of as being analogous to the boundary conditions associated with differential equations for instance. Let these new conditions be taken in the forms

$$(41)\quad \sum_{j=1}^{n+h} a_{ij}u_j = b_i, \quad \sum_{j=1}^{n+h} a_{ij}x_j = 0, \quad i = n+1, n+2, \cdots, n+s,\ s \leqq h.$$

The question is on the possibility of choice of the $c^{(k)}$ in (40) and of the $\bar{c}^{(k)}$ in the relation

$$x_i = \bar{c}^{(1)}x_i^{(1)} + \cdots + \bar{c}^{(h)}x_i^{(h)}$$

so that systems (41) shall be satisfied; that is, it is on the possibility of the systems

$$(42)\quad \sum_{k=1}^{h} c^{(k)} \sum_{j=1}^{n+h} a_{ij}x_j^{(k)} + \sum_{j=1}^{n+h} a_{ij}\bar{u}_j = b_i, \quad \sum_{k=1}^{h} \bar{c}^{(k)} \sum_{j=1}^{n+h} a_{ij}x_j^{(k)} = 0,$$
$$i = n+1, \cdots, n+s.$$

In case $s < h$ the latter system always has solutions $\bar{c}^{(k)}$. It may also have solutions when $s = h$. When it has solutions the former system may or may not have a solution; its having a solution depends on the character of the $\bar{u}_j$ and hence on the character of the b_i in (34) and also on the character of the new a_{ij} and b_i in (41). If $s = h$ and if the last system in (42) has no solution except that for which the $\bar{c}^{(k)}$ are all zero then the first system in (42) has a unique solution. This last algebraic theorem (essentially obvious in its character) has for an analogue a fundamental theorem in the theory of differential equations (one which is not obvious in character).*

Let us now consider the system

$$(43)\qquad \sum_{j=1}^{n} a_{ij}v_j = c_i, \qquad i = 1, 2, \cdots, n,$$

* Bôcher, "Les Méthodes de Sturm," pp. 19ff.

where the determinant Δ of the coefficients, $\Delta \equiv |a_{ij}|$, is different from zero. Then the customary solution (which is unique) may be written in the form

$$v_j = \sum_{i=1}^{n} G_{ij} c_i, \qquad j = 1, 2, \cdots, n, \tag{44}$$

where

$$G_{ij} = \frac{\Delta_{ij}}{\Delta},$$

Δ_{ij} being the cofactor of the element in the ith row and jth column of Δ. The function G_{ij} has been called the Green's function* for the homogeneous system $\Sigma a_{ij} v_j = 0$ corresponding to system (43). For fixed i it satisfies every equation of this system except the ith, the first member of the ith equation having for this value $v_j = \Delta_{ij}/\Delta$ the value 1 instead of zero. Thus, by means of the homogeneous system, *which is incompatible*, we can define a Green's function G_{ij} of such sort that the solution of (43) for every set c_i is expressible in the form (44). If the homogeneous system is not incompatible it is easy to see that no function G_{ij} exists having with respect to (43) the named property.

If H_{ij} is the Green's function of the system adjoint to system (43) in the sense of § 2 it is obvious that $H_{ji} = G_{ij}$.

This obvious matter for algebraic systems is mentioned here for its use in suggesting the requisite procedure for analogous properties of certain functional equations to be treated in later papers and especially for its use in transforming the expansions of § 4 into the form of contour integrals in § 10 immediately following.

10. *Condensation of Algebraic Expansions into Contour Integrals.* Let us consider systems (10) with a single parameter λ and suppose that the determinant $\Delta(\lambda)$ of the first system has n distinct simple roots $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}$ and that to each root corresponds a unque solution of the system (except for a constant factor). Denote by $\Delta_{ij}(\lambda)$ the cofactor of the element in the ith row and jth column of $\Delta(\lambda)$. Then the Green's functions $G_{ij}(\lambda)$ and $H_{ij}(\lambda)$ for the two systems in (10) are respectively

$$G_{ij}(\lambda) = \frac{\Delta_{ij}(\lambda)}{\Delta(\lambda)}, \qquad H_{ij}(\lambda) = G_{ji}(\lambda) = \frac{\Delta_{ji}(\lambda)}{\Delta(\lambda)}.$$

These Green's functions are analytic in the complex variable λ except at the zeros of $\Delta(\lambda)$, that is, except for characteristic values $\lambda^{(k)}$ of systems (10).

Now we may write the Green's function $G_{ij}(\lambda)$ in the forms

$$G_{ij}(\lambda) = \frac{R_{kij}}{\lambda - \lambda^{(k)}} + S_{kij}(\lambda), \qquad k = 1, 2, \cdots, n,$$

* Bôcher, *Annals of Mathematics*, (2) 13 (1911), pp. 71ff.

where $S_{kij}(\lambda)$ is analytic in λ at $\lambda = \lambda^{(k)}$ and R_{kij} is the residue at $\lambda = \lambda^{(k)}$ of $G_{ij}(\lambda)$ considered as a function of the complex variable λ. We propose to evaluate the residue R_{kij} in terms of the solutions of systems (10).

In the first place, we have

$$R_{kij} = \frac{\Delta_{ij}(\lambda^{(k)})}{\Delta'(\lambda^{(k)})}.$$

where $\Delta'(\lambda^{(k)})$ denotes the value at $\lambda = \lambda^{(k)}$ of the derivative of $\Delta(\lambda)$ with respect to λ. The quantities $x_j = R_{k\sigma j}$, for fixed k and σ, afford a solution of the first system in (10) for $\lambda = \lambda^{(k)}$. Likewise, the quantities $y_j = R_{kj\rho}$, for fixed k and ρ, afford a solution of the second system in (10) for $\lambda = \lambda^{(k)}$. But all solutions of these systems for $\lambda = \lambda^{(k)}$ are constant multiples of $x_j^{(k)}$ and $y_j^{(k)}$ respectively. Hence we have

$$R_{kij} = \gamma_k x_j^{(k)} y_i^{(k)},$$

where γ_k is independent of i and j. It remains to evaluate γ_k.

Now we have

$$\lim_{\lambda=\lambda^{(k)}} \{(\lambda - \lambda^{(k)})G_{it}(\lambda) - \gamma_k x_t^{(k)} y_i^{(k)}\} = 0;$$

whence it follows that

$$(45) \quad \lim_{\lambda=\lambda^{(k)}} \left\{ (\lambda - \lambda^{(k)}) \sum_{i=1}^{n} \sum_{j=1}^{n} G_{it}(\lambda) b_{ij} x_j^{(k)} - \gamma_k x_t^{(k)} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_j^{(k)} y_i^{(k)} \right\} = 0.$$

But

$$\sum_{j=1}^{n} (c_{ij} + \lambda b_{ij}) x_j^{(k)} = (\lambda - \lambda^{(k)}) \sum_{j=1}^{n} b_{ij} x_j^{(k)},$$

so that from the fundamental property of the Green's function we have

$$x_t^{(k)} = \sum_{i=1}^{n} G_{it}(\lambda)(\lambda - \lambda^{(k)}) \sum_{j=1}^{n} b_{ij} x_j^{(k)}.$$

Putting this value of $x_t^{(k)}$ into (45) we have readily

$$1 - \gamma_k \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_j^{(k)} y_i^{(k)} = 0.$$

With the value of γ_k afforded by this relation we have for the residue R_{kij} of $G_{ij}(\lambda)$ at $\lambda = \lambda^{(k)}$ in the complex plane the value

$$(46) \qquad R_{kij} = \frac{x_j^{(k)} y_i^{(k)}}{\sum_{\rho=1}^{n} \sum_{\sigma=1}^{n} b_{\rho\sigma} x_\sigma^{(k)} y_\rho^{(k)}}.$$

From this result we have for the kth term in the expansion of z_t in terms

of $x_t^{(k)}$ by (16) and (17) the value

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}z_i y_j^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}x_i^{(k)}y_j^{(k)}} x_t^{(k)} = \sum_{i=1}^{n}\sum_{j=1}^{n} R_{kjt}b_{ji}z_i = \frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_k}\sum_{i=1}^{n}\sum_{j=1}^{n} G_{jt}(\lambda)b_{ji}z_i d\lambda,$$

where Γ_k is a contour about $\lambda^{(k)}$ inclosing no other characteristic value of system (10). Then, if Γ is a contour inclosing all the characteristic values of systems (10), we have

$$(47)\qquad z_t = \frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\sum_{i=1}^{n}\sum_{j=1}^{n} G_{jt}(\lambda)b_{ji}z_i d\lambda, \qquad t = 1, 2, \cdots, n.$$

Thus we replace the expansion for z_t by a contour integral. It is clear that any set of terms of the expansion may likewise be replaced by the same integral taken about a contour inclosing appropriate corresponding zeros of $\Delta(\lambda)$.

Contour integrals of this character were first employed by Birkhoff.* They served admirably in the convergence proofs associated with the expansions of functions in terms of functions defined by differential systems, these systems being of such character as to be realized as limiting cases of the algebraic systems here in consideration. We shall find of fundamental value in our general investigation various limiting forms of the contour integrals developed in this section.

It is possible (but not important for our purposes) to modify and extend the foregoing analysis so as to avoid the restriction that the zeros of $\Delta(\lambda)$ shall be distinct.

Let us now pass to the more general systems (1) and (2). For any given fixed value of h let Δ_h $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the determinant in the first member of (3) and let Δ_{hij} $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the cofactor of the element in the ith row and the jth column of this determinant. Then the Green's function G_{hij} of the hth system in (1) has the form

$$G_{hij} = \frac{\Delta_{hij}(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\Delta_h(\lambda_1, \lambda_{,2} \cdots, \lambda_r)},$$

We assume for the present purpose that systems (1) [and also systems (2)] are so restricted that the solution of each for characteristic values of the λ's is unique (except for a constant factor) and that

$$\lim_{\lambda_1=\lambda_1^{(k)}, \cdots, \lambda_r=\lambda_r^{(k)}} \frac{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\lambda_h - \lambda_h^{(k)}}$$

* *Transactions of the American Mathematical Society,* 9 (1908): 373–395.

exists and is finite, where $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$ is any set of characteristic values for systems (1) and (2). Then the function

$$G_{hij}(\lambda_1^{(i)}, \cdots, \lambda_{h-1}^{(k)}, \lambda_h, \lambda_{h+1}^{(k)}, \cdots, \lambda_r^{(k)})$$

of the complex variable λ_h has a pole of the first order at the point $\lambda_h = \lambda_h^{(k)}$. Let its residue at this point be denoted by R_{khij}.

In order to evaluate the residue R_{khij} we apply the earlier result of this section for R_{kij} to the system which results from the hth system in (1) on replacing each λ_i except λ_h by the corresponding characteristic value $\lambda_i^{(k)}$. Thus we have

$$R_{hkij} = \gamma_{kh} x_h^{(k)} y_i^{(k)}$$

where γ_{kh} is independent of i and j. Then we have

$$\prod_{h=1}^{r} R_{khi_hj_h} = \gamma_k \prod_{h=1}^{r} x_{hj_h}^{(k)} y_{hi_h}^{(k)},$$

where γ_k is independent of the i and the j.

Then if we write

$$G_{i_1j_1 \cdots i_rj_r}(\lambda_1, \lambda_2, \cdots, \lambda_r) = \prod_{h=1}^{r} G_{hi_hj_h}(\lambda_1, \lambda_2, \cdots, \lambda_r)$$

we have

$$\lim_{\lambda_1=\lambda_1^{(k)}, \ldots, \lambda_r=\lambda_r^{(k)}} (\lambda_1 - \lambda_1^{(k)}) \cdots (\lambda_r - \lambda_r^{(k)}) G_{i_1j_1 \cdots i_rj_r}(\lambda_1, \cdots, \lambda_r) = \gamma_k \prod_{h=1}^{r} x_{hj_h}^{(k)} y_{hi_h}^{(k)};$$

whence it follows that

$$(48) \quad \lim_{\lambda_1=\lambda_1^{(k)}, \ldots, \lambda_r=\lambda_r^{(k)}} (\lambda_1 - \lambda_1^{(k)}) \cdots (\lambda_r - \lambda_r^{(k)})$$
$$\times \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} G_{i_1t_1 \cdots i_rt_r} D_{j_1i_1 \cdots j_ri_r} \prod_{h=1}^{r} x_{kj_h}^{(h)}$$
$$= \gamma_k \prod_{h=1}^{r} x_{ht_h}^{(k)} \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{j_1i_1 \cdots j_ri_r} \prod_{h=1}^{r} x_{hj_h}^{(k)} y_{hi_h}^{(k)}.$$

Employing the hth system in (1) and proceeding by the method used earlier in this section for evaluating $x_i^{(k)}$ in terms of the Green's function, we have

$$x_{ht_h}^{(k)} = \sum_{s=1}^{r} (\lambda_s - \lambda_s^{(k)}) \sum_{i_h=1}^{n_h} \sum_{j_h=1}^{n_h} a_{shi_hj_h} x_{hj_h}^{(k)} G_{hi_ht_h}(\lambda_1, \cdots, \lambda_r), \qquad h = 1, 2, \cdots, r.$$

From this we may form the product

$$x_{1t_1}^{(k)} x_{2t_2}^{(k)} \cdots x_{rt_r}^{(k)}$$

and simplify its expression by taking the limit for any convenient approach

of $\lambda_1, \cdots, \lambda_r$ to the characteristic values $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$. Then, in a certain range of cases (if not always) it turns out that the value of the product in question is that of the first member of equation (48). *We confine attention to the cases in which this new condition is realized.* Then we have

$$1 = \gamma_k \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{j_1 i_1 \ldots j_r i_r} \prod_{h=1}^{r} x_{h j_h}^{(k)} y_{h i_h}^{(k)}.$$

Thence it follows that

$$\lim_{\substack{\lambda_1=\lambda_1^{(k)} \\ \cdots \\ \lambda_r=\lambda_r^{(k)}}} G_{j_1 t_1 \ldots j_r t_r}(\lambda_1, \cdots, \lambda_r) = \frac{\prod_{h=1}^{r} x_{h t_h}^{(k)} y_{h j_h}^{(k)}}{\sum_{\rho_1=1}^{n_1} \sum_{\sigma_1=1}^{n_1} \cdots \sum_{\rho_r=1}^{n_r} \sum_{\sigma_r=1}^{n_r} D_{\sigma_1 \rho_1 \ldots \sigma_r \rho_r} \prod_{h=1}^{r} x_{h \sigma_h}^{(k)} y_{h \rho_h}^{(k)}}.$$

From this result we have for the kth term of the expansion of $z_{t_1 t_2 \ldots t_r}$ afforded by (14) the value

$$\frac{1}{(2\pi\sqrt{-1})^r} \int_{\Gamma_{1k}} \int_{\Gamma_{2k}} \cdots \int_{\Gamma_{rk}} \left\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1 j_1 \ldots i_r j_r} z_{i_1 i_2 \ldots i_r} G_{j_1 t_1 \ldots j_r t_r} \right\} d\lambda_1 d\lambda_2 \cdots d\lambda_r,$$

where Γ_{hk} $(h = 1, 2, \cdots, r)$ is a contour in the λ_h-plane about the point $\lambda_h^{(k)}$ and containing in its interior no other characteristic value of λ_h. If we replace the contour Γ_{hk} $(h = 1, 2, \cdots, r)$ by Γ_h, a contour which includes within it all the characteristic values of λ_h, and perform the same multiple integration about such contours, we shall have the value of the function $z_{t_1 t_2 \ldots t_r}$. It is clear that we may form similarly the contour integral for any given partial sum of the series for $z_{t_1 t_2 \ldots t_r}$ afforded by relation (14).

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ALGEBRAIC THEORY OF THE EXPRESSIBILITY OF CUBIC FORMS AS DETERMINANTS, WITH APPLICATION TO DIOPHANTINE ANALYSIS.

By L. E. Dickson.

1. It was proved geometrically by H. Schröter* and more simply by L. Cremona† that a sufficiently general cubic surface $f = 0$ is the locus of the intersections of corresponding planes of three projective bundles of planes:

$$\kappa l_{11} + \lambda l_{12} + \mu l_{13} = 0, \qquad \kappa l_{21} + \lambda l_{22} + \mu l_{23} = 0, \qquad \kappa l_{31} + \lambda l_{32} + \mu l_{33} = 0,$$

where κ, λ, μ are parameters and the l_{ij} are linear homogeneous functions of $x_1, \cdots, x_4$. Hence the surface is expressible in determinantal form $|l_{ij}| = 0$. Solving the three equations, we get

$$x_1 : x_2 : x_3 : x_4 = f_1 : f_2 : f_3 : f_4,$$

where f_i is a homogeneous cubic function of κ, λ, μ. We thus secure a parametric representation of the points of the surface, which is therefore unicursal.

In case all of the coefficients of the l_{ij} are rational, we have the complete solution in rational numbers of the Diophantine equation $f = 0$. However, there exist cubic equations $f = 0$ whose rational solutions involve three parameters homogeneously such that f is not expressible rationally in determinantal form (§ 12).

With the application to Diophantine analysis in mind, I here discuss algebraically the problem to express a given cubic form f as a determinant $|l_{ij}|$ of the third order. Since we are interested ultimately in the case in which the coefficients of the l_{ij} are rational and since three linear functions of four variables vanish for values not all zero of the variables, we shall assume that f vanishes at a known rational point. Then the coefficients of the l_{ij} are shown to be expressible rationally in terms of a root of an algebraic equation whose leading coefficient is not zero if $f = 0$ has no singular point. The existence of a rational root may be decided by a finite number of trials.

The corresponding binary problem is solved by the identity

* *Journal für Mathematik*, Vol. 62 (1863), p. 265.

† *Ibid.*, Vol. 68 (1868), p. 79. Cf. Clebsch, *ibid.*, 65 (1866), p. 359.

$$ax^3 + bx^2y + cxy^2 + dy^3 \equiv \begin{vmatrix} ax + by & y & 0 \\ -cy & x & y \\ dy & 0 & x \end{vmatrix}.$$

In the ternary case we obtain simple geometrical criteria that a cubic curve with a rational point be expressible rationally in determinantal form. There exist determinants of the third order which vanish for no rational point, for example, the determinant $\Delta(\xi)$ of the general number $\xi = x + y\theta + z\zeta$ of the algebra having the units 1, θ, $\zeta = \theta^2$, where θ is a root of a cubic with no rational root. But ternary forms without rational solutions have little interest in Diophantine analysis and will not be treated here.

Given one representation of a form as a determinant, we may derive an infinitude of representations by the familiar operations which leave a determinant unaltered in value. The more definitive problem treated here is to find a representative of each class of equivalent matrices with linear elements which have a given determinant. This raises the question of the number of classes of such matrices.

In another paper* I prove that every binary form, every ternary form, and every quaternary quadratic form are expressible in determinantal form; while, apart from these and the quaternary cubic, no further general form has this property. The rarety of such forms justifies the present investigation of cubic forms with attention to rationality. The theory covers every cubic form, not merely a sufficiently general one.

2. The given rational point on the locus can be transformed rationally into $(1, 0, \cdots, 0)$. Then the locus is $f = 0$, where $f = x^2f_1 + xf_2 + f_3$, where f_j is of degree j in $y, z, \cdots$. If f_1 is not identically zero, we take it as the new variable y. Then by adding to x a suitable linear function of $y, z, \cdots$, we may delete rationally from f_2 all terms with the factor y. Hence

$$f = x^2y + xf_2 + f_3, \tag{1}$$

where the quadratic f_2 is free of x and y, and the cubic f_3 is free of x. But if $f_1 \equiv 0$ and $f_2 \not\equiv 0$, we may transform f_2 into $ay^2 + q$, where $a \neq 0$ and q lacks y. By adding to x a suitable linear function of $y, z, \cdots$, we may delete rationally from f_3 the terms with the factor y^2 and obtain $f = x(ay^2 + q) + yr + s$, where q, r, s lack both x and y. Taking ax as a new x, we have $a = 1$. Interchanging x and y, we obtain a form of type (1).

3. Let the form (1) equal to a determinant of the third order whose elements are linear forms with rational coefficients. We may assume that x occurs in the first element of the first row and that its coefficient is unity.

* *Transactions Amer. Math. Soc.*, April, 1921.

The remaining elements in the first row and first column may be assumed to be free of x. By interchanges of the last two rows and last two columns, we may assume that the second element of the second row contains x; its coefficient may be made equal to unity. Hence we may take

$$f = \begin{vmatrix} x + l_1 & l_2 & l_3 \\ l_4 & x + l_5 & l_6 \\ l_7 & l_8 & y \end{vmatrix}, \tag{2}$$

where each l_i lacks x. After subtracting multiples of the last row and last column from the remaining rows and columns, respectively, we may assume that l_3, l_6, l_7, l_8 lack x and y. Thus by the terms linear in x,

$$l_1 + l_5 = 0, \qquad - l_3 l_7 - l_6 l_8 = f_2. \tag{3}$$

The interchange of the first two columns and the first two rows of (2) corresponds to the substitution

$$S = (l_1 l_5)(l_2 l_4)(l_3 l_6)(l_7 l_8).$$

The interchange of rows with columns corresponds to

$$T = (l_2 l_4)(l_3 l_7)(l_6 l_8).$$

Add the products of the elements of the second row by k to the elements of the first row and then subtract the products of the elements of the first column by k from the elements of the second column; we obtain a determinant of the same form with the same l_4, l_6, l_7 and with

$$\begin{gathered} l'_1 = l_1 + k l_4, \qquad l'_2 = l_2 - k l_1 + k l_5 - k^2 l_4, \qquad l'_5 = l_5 - k l_4, \\ l'_3 = l_3 + k l_6, \qquad l'_8 = l_8 - k l_7. \end{gathered} \tag{4}$$

Proceeding as before with the words first and second interchanged, we obtain a like determinant with the same l_2, l_3, l_8 and with

$$\begin{gathered} l'_1 = l_1 - k l_2, \qquad l'_4 = l_4 + k l_1 - k l_5 - k^2 l_2, \qquad l'_5 = l_5 + k l_2, \\ l'_6 = l_6 + k l_3, \qquad l'_7 = l_7 - k l_8. \end{gathered} \tag{5}$$

The last result may be obtained by transforming (4) by S.

4. For 3 variables (1) becomes

$$f = x^2 y + e x z^2 + C, \qquad C = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3. \tag{6}$$

The coefficient of x^3 in the Hessian of (6) is $- 8e$. Hence (1, 0, 0) is a point of inflexion of $f = 0$ if and only if $e = 0$. If $e \neq 0$, we multiply y by e^2 and x by $1/e$ and obtain a form of type (6) with $e = 1$. We seek the conditions under which (6) can be expressed as a determinant (2), where l_3, l_6, l_7, l_8 are multiples of z.

First, let either $l_3 \equiv l_6 \equiv 0$ or $l_7 \equiv l_8 \equiv 0$. Then (2) is the product of y by a quadratic which evidently vanishes when $x = -l_1$, $l_2 = 0$. Hence (6) must be the product of y by a ternary quadratic q which vanishes at a rational point. Conversely, the product of such a q by a rational linear form l can be transformed* rationally into one of

$$l(xy - kz^2) = \begin{vmatrix} l & 0 & 0 \\ 0 & x & z \\ 0 & kz & y \end{vmatrix}, \qquad l(ay^2 + byz + cz^2) = \begin{vmatrix} l & 0 & 0 \\ 0 & ay + bz & z \\ 0 & -cz & y \end{vmatrix}.$$

However, the product of a linear form, say y, by an arbitrary ternary quadratic form Q is expressible rationally as a determinant. If Q lacks x^2, it vanishes at (1, 0, 0). In the contrary case we may delete the terms in xy and xz by a transformation on x and use identity (7).

Second, let l_3, l_6, l_7, l_8 be not all zero identically. Since S, T and ST replace l_7 by l_8, l_3, l_6, we may take $l_7 \not\equiv 0$. Hence we may set $l_7 = z$. By (4), we may take $l_3 = 0$. By (3), $-l_3 z \equiv ez^2$, whence $l_3 = -ez$.

We first treat the case $e = 0$. Then $l_3 = l_8 = 0$, $l_7 = z$. Hence shall

$$C = \begin{vmatrix} l_1 & l_2 & 0 \\ l_4 & -l_1 & l_6 \\ z & 0 & y \end{vmatrix},$$

where l_6 is a multiple bz of z. If $l_2 = ay$, $C = y(-l_1^2 - ayl_4 + abz^2)$, which requires $\delta = 0$ in C and is then satisfied if $a = 1$, $l_1 \equiv 0$, $b = \gamma$, $l_4 = -\alpha y - \beta z$, so that we have the identity†

$$(7) \qquad y(x^2 + \alpha y^2 + \beta yz + \gamma z^2) = \begin{vmatrix} x & y & 0 \\ -\alpha y - \beta z & x & \gamma z \\ z & 0 & y \end{vmatrix}.$$

Next, let l_2 contain z, whose coefficient may be divided out of the second column of determinant C and multiplied into its second row. Then $l_2 = z + cy$. By (5), we may delete z from l_1 and set $l_1 = ay$. Write $l_4 = gy + hz$, $l_6 = tz$. Then shall

$$C = \begin{vmatrix} ay & z + cy & 0 \\ gy + hz & -ay & tz \\ z & 0 & y \end{vmatrix}.$$

$$= -(a^2 + cg)y^3 - (g + ch)y^2 z + (ct - h)yz^2 + tz^3,$$

whence

$$t = \delta, \qquad h = c\delta - \gamma, \qquad g = -ch - \beta, \qquad a^2 + cg = -\alpha.$$

The last relation furnishes the condition

* Dickson, "Algebraic Invariants," 1914, p. 24.

† If (6) with $e = 0$ has any linear factor, it has the factor y.

$$(8) \qquad -\alpha + \beta c - \gamma c^2 + \delta c^3 = a^2,$$

which is solvable in rational numbers if and only if (6) with $e = 0$ contains the rational point* $(a, 1, -c)$. The latter may be any point not on the inflexion tangent $y = 0$; if $\delta \neq 0$, it is any point except $(1, 0, 0)$.

Next, let $e = 1$, whence $l_3 = -z$. We have $l_6 = kz$. By (5), we may take $l_6 \equiv 0$. Thus (2) becomes

$$(9) \qquad \begin{vmatrix} x + l_1 & l_2 & -z \\ l_4 & x - l_1 & 0 \\ z & 0 & y \end{vmatrix} = x^2y + xz^2 - y(l_1^2 + l_2l_4) - z^2l_1.$$

Identify this with (6). By the terms in z^3, we see that $l_1 = ay - \delta z$. Set $l_2 = cy + dz$, $l_4 = gy + hz$. Then the conditions are

$$cg = -\alpha - a^2, \qquad dg + ch = 2a\delta - \beta, \qquad dh = -\gamma - \delta^2 - a.$$

Also, $dg - ch = 2r$ must be rational. From

$$(dg - ch)^2 \equiv (dg + ch)^2 - 4cg \cdot dh,$$

we obtain the condition that

$$(10) \qquad -4a^3 - 4a^2\gamma - 4a(\alpha + \beta\delta) + \beta^2 - 4\alpha(\gamma + \delta^2) = 4r^2$$

shall have rational solutions a, r. Then we have rational values of $dg = A$, $ch = B$, $dh = C$, $cg = D$ such that $AB = CD$. If $A \neq 0$ or $C \neq 0$, we may take $d = 1$, $g = A$, $h = C$, $c = D/A$ or B/C. If $B \neq 0$ or $D \neq 0$, we may take $c = 1$, $g = D$, $h = B$, $d = C/B$ or A/D.

To interpret (10), replace x by $l_1 = ay - \delta z$ in (6) with $e = 1$, and remove the factor y. We obtain a quadratic in $y : z$ with a rational root† if and only if (10) has rational solutions. Then (6) has three rational points of intersection with $x = ay - \delta z$ for some rational value of a, i.e., for some line other than $y = 0$ through $(-\delta, 0, 1)$. The latter point is the tangential of $P = (1, 0, 0)$, i.e., the new point in which the tangent $y = 0$ at P to the cubic curve meets the curve.

THEOREM. *Any reducible cubic curve is expressible rationally in determinantal form. An irreducible cubic curve with a rational inflexion point is expressible rationally in determinantal form if and only if it contains a further rational point. A cubic curve with a rational point P not an inflexion is expressible rationally in determinantal form if and only if it has three rational points of intersection with some line, other than the tangent at P, through the tangential of P.*

5. For four variables, we may set $f_2 = az^2 + bw^2$ in (1), where $b = 0$

* For which $x = l_1$, $l_2 = 0$, whence the elements of the second column of (2) are all zero.

† Which makes $l_2 = 0$ in accord with the second column of (9).

if $a = 0$. First, let l_3, l_6, l_7, l_8 be not all free of z, the contrary case being treated in § 9. In view of substitutions S and T of § 3, we may assume that l_7 contains z; its coefficient may be made unity by removal of factors from the last row and last column of (2). By (4), we may assume that $l_8 = vw$. We here take $v \neq 0$ (treating $v = 0$ in § 10). By dividing the elements of the second column by v and multiplying those of the second row by v, we have $l_8 = w$. Then by (5) we may take $l_7 = z$. Hence,* by (3),

$$l_3 = -az + gw, \qquad l_6 = -gz - bw,$$

$$(11) \quad f = \begin{vmatrix} x + l_1 & l_2 & l_3 \\ l_4 & x - l_1 & l_6 \\ z & w & y \end{vmatrix} = x^2y + x(az^2 + bw^2) + \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & -l_1 & l_6 \\ z & w & y \end{vmatrix}.$$

Set

$$(12) \quad l_1 = cy + dz + ew, \qquad l_2 = hy + jz + kw, \qquad l_4 = my + nz + pw,$$

$$(13) \quad f_3 = Ay^3 + By^2z + Cy^2w + Dyz^2 + Eyzw + Fyw^2 + Gz^3 + Hz^2w + Jzw^2 + Lw^3.$$

Then (11) is identical with (1) if and only if

$$-c^2 - hm = A, \quad -2cd - hn - jm = B, \quad -2ce - hp - km = C,$$

$$-d^2 - jn - gh - ac = D, \quad -2de - jp - kn + 2gc - am - bh = E,$$

$$(14) \quad -e^2 - kp + bc + gm = F, \qquad -ad - gj = G,$$

$$2gd - an - bj - gk - ae = H,$$

$$bd + gn + 2ge - ap - bk = J, \qquad be + gp = L.$$

We shall designate these equations as (A), $\cdots$, (L).

First, let $a = b = 0$. If $g = 0$, then $l_3 \equiv l_6 \equiv 0$ and (11) is the product of y by its minor. Postponing this case to the end of § 11, we have $g \neq 0$. Let $dg = \delta$, $eg = \epsilon$. Then equations (L), $\cdots$, (D) give

$$gp = L, \quad gj = -G, \quad gn = J - 2\epsilon, \quad gk = 2\delta - H, \quad g^3m = \epsilon^2 + g^2F + L(2\delta - H),$$

$$g^3h = -\delta^2 - g^2D + G(J - 2\epsilon), \qquad 2g^3c = g^2E - LG - HJ + 2\epsilon H + 2\delta J - 2\delta\epsilon.$$

Inserting these values into the product of (B) and (C) by g^4, we see that the terms of the third degree in δ and ϵ cancel, giving

$$(15) \quad \delta^2 J + 2\delta\epsilon H + 3\epsilon^2 G + \delta(g^2E - 3LG - HJ) + \epsilon(2g^2D - 4GJ) + g^4B - g^2(FG + DJ) + GJ^2 + GHL = 0,$$

$$(16) \quad 3\delta^2 L + 2\delta\epsilon J + \epsilon^2 H + \delta(2g^2F - 4HL) + \epsilon(g^2E - 3GL - HJ) + g^4C - g^2(FH + DL) + H^2L + GJL = 0.$$

* It is proved in § 13 that no further normalization of our determinant is possible and that the two excluded cases are truly exceptional.

Multiplying (A) by $3g^6$, we obtain an equation whose terms of highest degree are $-6\delta^3L-6\delta^2\epsilon J-6\delta\epsilon^2H-6\epsilon^3G$. Hence by adding to it the product of (15) by 2ϵ and the product of (16) by 2δ, we obtain the quadratic

$$(17)\qquad \begin{aligned}&\delta^2(g^2F-5HL+3J^2)+\epsilon^2(g^2D-5GJ+3H^2)+\delta\epsilon(g^2E-21GL+5HJ)\\&+\delta[2g^4C+g^2(3EJ-2FH-8DL)+5GJL+2H^2L-3J^2H]\\&+\epsilon[2g^4B+g^2(3EH-2DJ-8FG)+5GHL+2GJ^2-3H^2J]\\&+3g^6A+\tfrac{3}{4}(g^2E-LG-HJ)^2-3(g^2D-GJ)(g^2F-HL)=0.\end{aligned}$$

The true resultant of three ternary quadratic forms u, v, w in x, y, z is known* to be the determinant of the coefficients of x^2, xy, xz, y^2, yz, z^2 in u, v, w, j_x, j_y, j_z, where j is the Jacobian (functional determinant) of u, v, w, while j_x is its partial derivative with respect to x. The resultant is of the fourth degree in the coefficients of each form u, v, w. To find the degree in g of the resultant R of our forms (15)–(17), we shall determine the terms of R of highest degree in g. Writing the functions (15)–(17) homogeneously in δ, ϵ, τ and retaining in each coefficient only the term of highest degree in g, we have

$$\begin{aligned}&\delta^2J+2\delta\epsilon H+3\epsilon^2G+\delta\tau g^2E+2\epsilon\tau g^2D+\tau^2g^4B,\\&3\delta^2L+2\delta\epsilon J+\epsilon^2H+2\delta\tau g^2F+\epsilon\tau g^2E+\tau^2g^4C,\\&g^2(\delta^2F+\delta\epsilon E+\epsilon^2D+2\delta\tau g^2C+2\epsilon\tau g^2B+3\tau^2g^4A).\end{aligned}$$

Thus $R=(g^2)^4r$, where r is the resultant when the factor g^2 is omitted from the last form. We make the transformation of variables $\delta=w$, $\epsilon=z$, $g^2\tau=y$ and obtain the partial derivatives of f_3, given by (13), with respect to z, w, y, respectively. Hence the resultant of the final forms is the discriminant Δ of f_3. Let J denote the Jacobian of these derivatives, and j denote the Jacobian of our forms having the resultant r. By a general theorem, j equals the product of J by the Jacobian g^2 of w, z, y with respect to δ, ϵ, τ. Now J is independent of g^2. Hence the exponent of $M=g^2$ in any coefficient of j exceeds by unity the exponent of τ in the term. This is therefore true also of the derivatives j_δ, j_ϵ, while in j_τ the exponent of M in any term exceeds by 2 the exponent of τ. We remove the factors M, M, M^2 from the last three functions. We now have six quadratic forms in δ, ϵ, τ, such that M does not occur as a factor of a coefficient of δ^2, $\delta\epsilon$, ϵ^2, while M (but not M^2) is a factor of the coefficients of $\delta\tau$ and $\epsilon\tau$, and M^2 (but not M^3) is a factor of the coefficient of τ^2. Their determinant thus has the factor M^4, but not M^5. Hence $r=M^4\cdot M^4\Delta$. But $R=M^4r$. Hence $R=M^{12}\Delta$. Thus g^2 is a root of an equation of degree 12 whose leading coefficient is the discriminant of f_3.

* Salmon, Algebra, § 90.

The constant term of this equation is not identically zero. It is in fact not zero when $J = H = 0$, $LG \neq 0$. For, then equations (15)–(17) become for $g = 0$

$$3G(\epsilon^2 - \delta L) = 0, \qquad 3L(\delta^2 - \epsilon G) = 0, \qquad 3LG(-7\delta\epsilon + \tfrac{1}{4}LG) = 0.$$

By the first two, $\delta\epsilon = 0$ or LG. Hence the resultant is not zero.

The coefficients in (15)–(17) are independent of g if and only if A, B, C, D, E, F are all zero. If they are zero the resultant is free of g and, as just shown, is not zero if $J = H = 0$, $LG \neq 0$.

THEOREM. *The problem to express $x^2y + f_3$ rationally in determinantal form, where f_3 is a cubic form in y, z, w, depends completely upon the determination of a rational square which satisfies an equation of degree* 12 *whose leading coefficient is the discriminant of f_3 and whose constant term is not zero identically. The equation reduces to its constant term if and only if f_3 lacks y. In particular, if $LG \neq 0$, $x^2y + Gz^3 + Lw^3$ is not expressible in determinantal form under our present assumptions* (cf. § 10).

The equation for g involves only even powers since this is true of (15)–(17). A more fundamental reason is given in § 14, where there occurs an example in which the equation for g^2 is a cubic, the discriminant of f_3 being zero.

To give another example, take $l_4 \equiv w$, $l_2 \equiv -y$, $l_1 \equiv 0$, $g = 1$. Then

$$\begin{vmatrix} x & -y & w \\ w & x & -z \\ z & w & y \end{vmatrix} = x^2y + w^3 + wy^2 + yz^2,$$

with no rational singular point. Since $C = D = L = 1$ and the remaining coefficients of f_3 are zero, (15)–(17) become

$$2g^2\epsilon = 0, \qquad 3\delta^2 = g^2 - g^4, \qquad g^2\epsilon^2 + (2g^4 - 8g^2)\delta = 0.$$

We desire that $g \neq 0$. The only real solution is $\delta = \epsilon = 0$, $g^2 = 1$. Hence the only rational representations as a determinant of type (11) are the above determinant and that obtained by changing the signs of the elements other than x in the first two rows.

6. We now discard the assumption that $a = b = 0$, but assume that $g \neq 0$, $\Delta \equiv g^2 + ab \neq 0$. It will prove convenient to introduce $d_1 = \Delta d$, $e_1 = \Delta e$ in place of d and e. The last four equations (14) give

$$g\Delta p = \Delta L - be_1, \qquad g\Delta j = -\Delta G - ad_1,$$
$$g\Delta k = 2gd_1 + ae_1 + \delta, \qquad g\Delta n = bd_1 - 2ge_1 + \epsilon,$$

where

$$\delta = -g^2H + g(bG - aJ) - a^2L, \qquad \epsilon = g^2J + g(aL - bH) + b^2G.$$

In (D), (E), (F), the determinant of the coefficients of h, m, c is $2g\Delta$. Hence

$$2g\Delta h = \alpha N + \beta ag - \gamma a^2, \qquad \alpha \equiv -D - d^2 - jn, \qquad N \equiv ab + 2g^2,$$
$$2g\Delta m = -\alpha b^2 + \beta bg + \gamma N, \qquad \beta \equiv -E - 2de - jp - kn,$$
$$2g\Delta c = \alpha bg - \beta g^2 + \gamma ag, \qquad \gamma \equiv F + e^2 + kp.$$

Replacing p, j, k, n by their values, we get

$$g^2\Delta^2\alpha = d_1^2(ab - g^2) - 2d_1e_1ag + d_1(\Delta bG + a\epsilon) - 2e_1\Delta Gg + \Delta G\epsilon - \Delta^2 Dg^2,$$
$$g^2\Delta^2\beta = -2d_1^2bg + 2d_1e_1(g^2 - ab) + 2e_1^2ag + d_1(\Delta La - b\delta - 2g\epsilon)$$
$$+ e_1(2g\delta - a\epsilon - \Delta Gb) - \delta\epsilon - \Delta^2Eg^2 + \Delta^2LG,$$
$$g^2\Delta^2\gamma = e_1^2(g^2 - ab) - 2d_1e_1bg + 2d_1\Delta Lg + e_1(\Delta La - b\delta) + \Delta L\delta + \Delta^2Fg^2.$$

Mulitplying equation (B) by $2g^4\Delta^4$ and inserting the preceding values of h, m, c, n, j, we obtain

$$(18) \quad 2d_1^3\Delta ab^2 - 2d_1^2e_1\Delta abg + 2d_1e_1^2\Delta a^2b - 2e_1^3\Delta a^2g$$
$$+ d_1^2\{\Delta(5Gb^2g^2 + 3Gab^3 - 2Lag^3 - 4La^2bg + 3ab\epsilon) + 2g^3(b\delta + g\epsilon)\}$$
$$+ 2d_1e_1\{(\Delta La^2 - \delta N)(g^2 - ab) - 2\Delta GbgN - 2a^2bg\epsilon\}$$
$$+ e_1^2\{2\Delta La^3g + (\Delta GN + \epsilon a^2)(ab + 3g^2) - 2\delta gaN\}$$
$$+ d_1\{-3\Delta^2LGgN - \Delta La^2(g\epsilon + 2b\delta) + 2\Delta^2Eg^5 - 2\Delta^2Fa^2bg^2$$
$$- 2\Delta^2Dbg^2N + \Delta^2G^2b^3 + 4\Delta Gb\epsilon N + \Delta Gb^2g\delta + a^2b\epsilon^2$$
$$+ (2g^3 - abg)\delta\epsilon\}$$
$$+ e_1\{(a\delta\epsilon - \Delta^2LGa)(ab + 4g^2) + L\Delta a^2(2g\delta - a\epsilon) + 2F\Delta^2g^3a^2$$
$$+ 2\Delta^2Eag^4 + 2\Delta^2Dg^3N - \Delta^2G^2b^2g - 4\Delta Gg\epsilon N + \Delta Gab^2\delta - a^2g\epsilon^2\}$$
$$+ 2\Delta^4g^4B + (\Delta^2Gb^2 + \Delta\epsilon N)(G\epsilon - \Delta Dg^2)$$
$$+ (ag\epsilon - \Delta Gbg)(\Delta^2LG - \Delta^2Eg^2 - \delta\epsilon)$$
$$- (\Delta^2GN + \Delta a^2\epsilon)(L\delta + \Delta Fg^2) = 0.$$

Similarly, from (C) we obtain an equation

$$(19) \qquad -2d_1^3\Delta b^2g - 2d_1^2e_1\Delta ab^2 - 2d_1e_1^2\Delta abg - 2e_1^3\Delta a^2b + \cdots,$$

which may be deduced from (18) by the substitution*

$$(20) \qquad (BC)(DF)(GL)(HJ)(ab)(d_1, -e_1)(g, -g),$$

which leaves A, E, Δ, N, β unaltered and induces

* $(d, -e)$ replaces d by $-e$ and e by $-d$.

$$(\delta, -\epsilon)(\alpha, -\gamma)(c, -c)(hm)(kn)(jp).$$

These substitutions and (zw) interchange l_2 with l_4, l_3 with l_6, and change the sign of l_1 and hence interchange the first two rows and first two columns of the initial determinant (11).

There remains the first equation (A) in (14). Its product by $4g^6\Delta^5$ may be written in the form

$$4g^6\Delta^5A - g^4\Delta^4(\alpha b - \gamma a)^2 + g^6\Delta^4(\beta^2 + 4\alpha\gamma) = 0. \tag{21}$$

Its terms of the highest degree in d_1, e_1 are $(3g^2 - ab)\lambda$, where

$$\lambda = \Delta(d_1^4b^2 + 2d_1^2e_1^2ab + e_1^4a^2).$$

The terms of the fourth degree in $bd_1(18) - ae_1(19)$ are $2ab\lambda$; those in $ge_1(18) + gd_1(19)$ are $-2g^2\lambda$. Hence by adding multiples of (18) and (19) to (21), we may cancel the terms of the fourth degree. Hence we have three cubic functions of d_1, e_1, whose true resultant R is known to be expressible as a determinant.

We proceed to find the terms of R of maximum degree in g. From the aggregate of terms in (18) which multiply each $d_1^re_1^s$ we omit all terms not of the highest degree in g, remove the common factor 2 and obtain

$$d_1^3g^2ab^2 - d_1^2e_1g^3ab + d_1e_1^2g^2a^2b - e_1^3g^3a^2 + d_1^2g^6J + 2d_1e_1g^6H + 3e_1^2g^6G + d_1g^9E + 2e_1g^9D + g^{12}B. \tag{22}$$

Applying to this the substitution (20), we obtain

$$- d_1^3g^3b^2 - d_1^2e_1g^2ab^2 - d_1e_1^2g^3ab - e_1^3g^2a^2b + 3d_1^2g^6L + 2d_1e_1g^6J + e_1^2g^6H + 2d_1g^9F + e_1g^9E + g^{12}C. \tag{23}$$

The resultant of these two functions and a third function T of d_1, e_1 is the same as the resultant of their sum and difference and T. Hence we may omit the terms involving only g^2. Remove the factors g^3, make the functions homogeneous in d_1, e_1, τ and set $g^3\tau = \phi$; we get the forms free of g:

$$- d_1^2e_1ab - e_1^3a^2 + d_1^2\phi J + 2d_1e_1\phi H + 3e_1^2\phi G + d_1\phi^2E + 2e_1\phi^2D + \phi^3B, \tag{24}$$

$$- d_1^3b^2 - d_1e_1^2ab + 3d_1^2\phi L + 2d_1e_1\phi J + e_1^2\phi H + 2d_1\phi^2F + e_1\phi^2E + \phi^3C. \tag{25}$$

In (21) the terms of maximum degrees in g are

$$3d_1^4g^4b^2 + 6d_1^2e_1^2g^4ab + 3e_1^4g^4a^2 - 8d_1^3g^7L - 8d_1^2e_1g^7J - 8d_1e_1^2g^7H - 8e_1^3g^7G - 4d_1^2g^{10}F - 4d_1e_1g^{10}E - 4e_1^2g^{10}D + d_1g^{11}(4JE - 8LD) + e_1g^{11}(4HE - 8GF) + 4g^{16}A. \tag{26}$$

The terms in g^{11} near the end do not contribute to the part of the resultant

of highest degree in g, since by omitting them (or replacing them by g^{13} with the coefficient zero), we shall obtain a resultant not identically zero. Also divide by $4g^4$, make homogeneous in d_1, e_1, τ, and set $g^3\tau = \phi$ as before; we get the form free of g:

$$(27)\quad \tfrac{3}{4}d_1^4b^2 + \tfrac{3}{2}d_1^2e_1^2ab + \tfrac{3}{4}e_1^4a^2 - 2d_1^3\phi L - 2d_1^2e_1\phi J - 2d_1e_1^2\phi H - 2e_1^3\phi G$$
$$- d_1^2\phi^2F - d_1e_1\phi^2E - e_1^2\phi^2D + \phi^4A.$$

Now (24) and (25) are the partial derivatives with respect to e_1 and d_1 of

$$(28)\quad A\phi^4 + Be_1\phi^3 + Cd_1\phi^3 + De_1^2\phi^2 + Ed_1e_1\phi^2 + Fd_1^2\phi^2 + Ge_1^3\phi + Hd_1e_1^2\phi$$
$$+ Jd_1^2e_1\phi + Ld_1^3\phi - \tfrac{1}{4}d_1^4b^2 - \tfrac{1}{2}d_1^2e_1^2ab - \tfrac{1}{4}e_1^4a^2$$
$$\equiv \phi f_3(\phi, e_1, d_1) - \tfrac{1}{4}(d_1^2b + e_1^2a)^2,$$

where $f_3(y, z, w)$ is given by (13). To (27) add the product of (24) by e_1 and the product of (25) by d_1; we get (28). Hence the resultant of (24), (25), (27) equals the resultant of (28) and its partial derivatives, i.e., the discriminant of (28). This discriminant is not zero for all values of $A, \cdots, L, a, b$. In fact, the general quartic curve becomes $zK - x^2y^2 = 0$ when referred to a triangle of reference whose side $z = 0$ is a bitangent and whose sides $x = 0$ and $y = 0$ are any lines through its two points of contact. Evidently (28) is of this form if x, y are the factors of $\frac{1}{2}(d_1^2b + e_1^2a)$.

The quartic curve (28) is a plane section* of the tangent cone to the cubic surface (1), viz.,

$$(1')\qquad x^2\phi + x(ae_1^2 + bd_1^2) + f_3(\phi, e_1, d_1) = 0$$

whose vertex is $\phi = d_1 = e_1 = 0$, $x = 1$. A point on (1′) will be a singular point if the partial derivatives with respect to x, e_1, d_1 vanish:

$$2x\phi + ae_1^2 + bd_1^2, \qquad 2axe_1 + \frac{\partial f_3}{\partial e_1}, \qquad 2xbd_1 + \frac{\partial f_3}{\partial d_1}.$$

Let $\phi \neq 0$ and substitute the value of x for which the first function vanishes into the others and (1′), and multiply each result by ϕ; we get

$$(29)\qquad \begin{gathered} - ae_1(ae_1^2 + bd_1^2) + \phi\frac{\partial f_3}{\partial e_1}, \qquad - bd_1(ae_1^2 + bd_1^2) + \phi\frac{\partial f_3}{\partial d_1}, \\ - \tfrac{1}{4}(ae_1^2 + bd_1^2)^2 + \phi f_3, \end{gathered}$$

which equal functions (24), (25), (28), respectively. Since a general cubic surface is reducible to (1′) by § 2, and has no singular point, we again conclude that the resultant of (24), (25), (27) is not identically zero.

* Miller, Blichfeldt, Dickson, Finite Groups, 1916, p. 352.

Multiply the first function (29) by e_1 and the second by d_1, add and apply Euler's theorem $\Sigma e_1 \partial f_3/\partial e_1 = 3f_3$; we get

$$e_1(24) + d_1(25) = -(ae_1^2 + bd_1^2)^2 + \phi\left(3f_3 - \phi\frac{\partial f_3}{\partial\phi}\right).$$

Subtract this from 4 times $e_1(24) + d_1(25) + (27) = (28)$, given above. Hence

$$3e_1(24) + 3d_1(25) + 4(27) = \phi\frac{\partial(\phi f_3)}{\partial\phi}.$$

Hence we may replace (27) by the cubic function $\partial(\phi f_3)/\partial\phi$. This may therefore be obtained by deleting the factor g^4 from

$$3e_1 g(22) + 3d_1 g(23) + (26),$$

after the three functions are abridged as above by omission of terms in g^2 and g^{11}. We also deleted the factor g^3 from each of the abridged cubics (22), (23). But the resultant is of the ninth degree in the coefficients of each form, and hence equals the product of $(g^3)^9(g^3)^9(g^4)^9 = g^{90}$ by the resultant r of the three cubic forms homogeneous in d_1, e_1, τ. Their Jacobian j equals the product of the Jabocian J of the equivalent forms in d_1, e_1, $\phi = g^3\tau$, by the Jacobian g^3 of d_1, e_1, ϕ with respect to d_1, e_1, τ. But J is independent of g. Hence the exponent of $M = g^3$ in any coefficient in j exceeds by unity the exponent of τ in the term. The same is therefore true of the derivatives j_{d_1} and j_{e_1}, while in j_ϕ the exponent of M exceeds by 2 the exponent of τ in any term. We remove the factors M, M, M^2 and obtain three quintic functions in which the exponent of M in any coefficient equals the exponent of τ in the term. The same is true of each of our three cubic forms. If we multiply each of them by d_1^2, d_1e_1, e_1^2, $d_1\tau$, $e_1\tau$, τ^2, we obtain 18 quintic forms. The determinant (of order 21) of the coefficients in these and the former three quintics is known* to be the resultant ρ of the three cubics. In each of the three equations obtained by use of the multiplier τ^2, the exponent of M in any term is 2 less than the exponent of τ; we multiply these equations by M^2. Similarly we multiply by M each of the six equations obtained by use of the multipliers $d_1\tau$, $e_1\tau$. Now in all 21 equations the exponent of M in any term equals the exponent of τ. Hence in the new determinant the elements of any column contain the same power of M as a factor, viz., the exponent of τ in the corresponding term $d_1^l e_1^m \tau^n$, $l + m + n = 5$. Hence the total exponent of the power of M which divides the determinant is $5 + 4(2) + 3(3) + 2(4) + 1(5) = 35$. But our new determinant equals $\rho(M^2)^3 M^6$. Hence ρ is the product of $M^{35}/M^{12} = M^{23}$ by a constant $\lambda \neq 0$. Thus, accounting for M^4 removed

* Salmon, Algebra, § 90.

above, $r = M^{27}\lambda$. Hence the resultant of our initial equations is of degree $90 + 3\cdot 27 = 171$ in g.

THEOREM. *When the quadratic form f_2 in z, w does not have rational factors, the problem to express $x^2y + xf_2 + f_3$ rationally in determinantal form depends completely upon the solution of an equation of degree 171 whose leading coefficient is not zero if the surface has no singular point.*

Perhaps this equation is reducible since its degree is high in comparison with the degree of the equation upon which depends the determination of the 27 ruled lines on the surface.

Material simplifications arise if $G = H = J = L = 0$, whence y is a factor of f_3, and $y = 0$ cuts the surface in ruled lines. Since $\delta = \epsilon = 0$, (18) and (19) have no quadratic terms. Dividing $bd_1(18) - ae_1(19)$ and $e_1(18) + d_1(19)$ by 2, we get

$$\begin{aligned}(30)\quad & \Delta ab(d_1^2b + e_1^2a)^2 + d_1^2\Delta^2g^2(Eg^3b - Fa^2b^2 - Db^2N)\\ & - e_1^2\Delta^2g^2(Eg^3a + Fa^2N + Da^2b^2) + 2d_1e_1\Delta^2g^4(Eab - Fag + Dbg)\\ & \qquad + \Delta^4g^4(d_1Bb - e_1Ca) = 0,\end{aligned}$$

$$\begin{aligned}(31)\quad & -\Delta g(d_1^2b + e_1^2a)^2 + d_1^2\Delta^2g^3(-Egb + FN + Db^2)\\ & + e_1^2\Delta^2g^3(Ega + Fa^2 + DN) + 2d_1e_1\Delta^2g^4(Eg + Fa - Db)\\ & \qquad + \Delta^4g^4(d_1C + e_1B) = 0.\end{aligned}$$

Dividing $g(30) + ab(31)$ by Δ^2g^4, we get

$$\begin{aligned}(32)\quad & (bd_1^2 - ae_1^2)[E(g^2 - ab) + 2Fga - 2Dbg]\\ & + 2d_1e_1[2Eabg + (g^2 - ab)(Db - Fa)] + \Delta^2bd_1(gB + Ca)\\ & \qquad + \Delta^2ae_1(Bb - Cg) = 0.\end{aligned}$$

We obtain a second quadratic from $[(21) + (30) + 3g(31)]/\Delta^2g^2$:

$$\begin{aligned}(33)\quad & d_1^2[2Ebg^3+F\rho+D(ab^3-g^2b^2)]+e_1^2[-2Eag^3+F(a^3b-g^2a^2)+D\rho]\\ & +2d_1e_1g^2[E(g^2+3ab)-2Fag+2Dbg]+d_1\Delta^2g^2(3Cg+Bb)\\ & \qquad +e_1\Delta^2g^2(3Bg-Ca)+\Delta^2g^4(E^2-4FD+4\Delta A)-\Delta^2g^2(Db+Fa)^2=0,\end{aligned}$$

where $\rho = a^2b^2 + 5abg^2 + 2g^4$. Here the coefficients of d_1^2 and e_1^2 will be proportional to b and $-a$, as in (32), if and only if $aF + bD = 0$, and then the same fact holds in (30) and (31). If also $B = C = 0$, no first degree terms in d_1, e_1 occur in our equations. Then (32) and (33) give

$$(bd_1^2-ae_1^2)Q=4\Delta[Eabg-(g^2-ab)Fa]Sg^2,\quad d_1e_1Q=\Delta[-E(g^2-ab)-4Fag]Sg^2,$$

$$Q = (g^2 - 3ab)t, \qquad t = E^2 + 4F^2a/b, \qquad S = 2\Delta A + \tfrac{1}{2}t.$$

From the squares of these, we get (if* $t \neq 0$)

$$(bd_1^2 + ae_1^2)^2(g^2 - 3ab)Q = 4ab\Delta^4 g^4 S^2.$$

We obtain a second expression for the left member by use of $3(30) + g(31)$:

$$-(bd_1^2 + ae_1^2)^2(g^2 - 3ab)\Delta + (bd_1^2 - ae_1^2)\Delta^2 g^4\left[2Eg + \frac{F}{b}(6ab + 2g^2)\right]$$
$$+ d_1e_1\Delta^2 g^3[2E(g^3 + 3abg) - 8Fag^2] = 0.$$

Multiplying this by Q, inserting the preceding values, and cancelling the common factor $-2\Delta^4 g^4$, we get either $S = 0$ or

$$g^4(4abA + t) + 2abg^2(4abA - t) + a^2b^2(4abA + t) = 0,$$

whence

$$\{g^2(4abA + t) + ab(4abA - t)\}^2 = -16a^3b^3At.$$

Hence the number of rational values of g^2 is 3 or 1 according as $-abAt$ is or is not a rational square. If also $A = D = -1$, $E = 2$, $F = 1$, $a = b = 1$, then $g = \pm 1, \pm 1 + \sqrt{2}, \pm 1 - \sqrt{2}$, and the surface has no singular point.

7. In §§ 7–11 we shall treat the special cases which were excluded in §§ 5–6. In each case the determinantal surface has a known ruled line whose equations are so simple that its occurrence on the given surface can be detected by inspection. When the given surface has the line as a ruling, its representation as a determinant is a much simpler problem than that treated in §§ 5–6. Accordingly one should first ascertain whether or not the given surface falls under one of these special cases.

Consider the case $g^2 + ab = 0$ which was excluded in § 6. If $a = 0$, then $b = 0$ (§ 5) and $g = 0$, $l_3 = l_6 = 0$, so that (11) is the product of y by its minor, a case treated at the end of § 11. Hence let $a \neq 0$. We first treat the case $b \neq 0$. Then

$$az^2 + bw^2 = bZW, \quad Z = w + \frac{a}{g}z, \quad W = w - \frac{a}{g}z, \quad l_3 = gW, \quad l_6 = -bW.$$

Introduce Z and W as new variables in place of z and w and let the new f_3 be given the same notation (13). We have therefore to consider a form (1) in which $f_2 = bzw$. Interchanging z and w if necessary, we may assume that, in (2), l_7 contains z. Proceeding exactly as in § 5, we may set $l_8 = w$, $l_7 = z$. By (3), $l_6 = tz$, $l_3 = (1 - t)w$. But l_3 and l_6 must be dependent and hence one of them zero. Applying substitution S, we may take $l_6 \equiv 0$.

* If $t = 0$, either $Sg^2 = 0$ or $E = F = 0$, since the determinant of the coefficients of E and F equals $-\Delta^2$. For $E = F = B = C = 0$, $D = 0$, (33) gives $A = 0$, and f has the factor x.

Thus we have (11) with $l_6 \equiv 0$, $l_3 = w$. It remains to identify the final determinant in (11) with f_3. Set $l_1 = cy + \lambda_1$, $l_4 = my + \lambda_4$. By the terms free of y,

$$w(w\lambda_4 + z\lambda_1) \equiv Gz^3 + Hz^2w + Jzw^2 + Lw^3,$$

whence $G = 0$, $\lambda_4 = Lw + nz$, $\lambda_1 = Hz + (J - n)w$. From the remaining terms we remove the factor y and have

$$-l_1^2 - l_2l_4 + w(wm + zc) \equiv Ay^2 + Byz + Cyw + Dz^2 + Ezw + Fw^2.$$

The resulting conditions $(A), \cdots, (F)$ are

$$-c^2 - mh = A, \quad -2cH - mj - nh = B, \quad -2c(J-n) - mk - Lh = C,$$

$$-H^2 - nj = D, \quad c - 2H(J-n) - nk - Lj = E, \quad m - Lk - (J-n)^2 = F.$$

First, let $n \neq 0$. From (D), (E), (F), (B),

$$j = \frac{-N}{n}, \quad k = \frac{P + cn}{n^2}, \quad m = \frac{Lc}{n} + \frac{R}{n^2}, \quad h = \frac{-B-2cH}{n} + \frac{N}{n^2}\left(\frac{Lc}{n} + \frac{R}{n^2}\right),$$

where

$$N = D + H^2, \qquad P = LN - n(E + 2HJ) + 2Hn^2,$$
$$R = n^2F + (J-n)^2n^2 + LP.$$

To clear the denominators of n multiply (A) by n^6, (C) by n^4, and use the abbreviations

$$N = D + H^2, \qquad S = E + 2HJ, \qquad T = F + J^2 + 2LH,$$
$$R = n^4 - 2n^3J + n^2T - nLS + L^2N.$$

Hence

$$(34) \quad c^2(n^6 - 2n^4LH + n^2L^2N) + c[2R(nLN - n^3H) - n^4LB] + R^2N - Rn^3B + n^6A = 0,$$

$$(35) \quad c^2n^2L + c(-n^5 + n^3T - 2n^2LS + 3nL^2N) + R(2n^2H - nS + 2LN) - n^3LB + n^4C = 0.$$

Retaining in each coefficient only the highest power of n, we get

$$c^2n^6 - 2cn^7H + n^8N = 0, \qquad c^2n^2L - cn^5 + 2n^6H = 0,$$

whose resultant is $n^{24}N$. Hence n satisfies an equation of degree 24.

Second, let $n = 0$, $L \neq 0$. This case occurs only if $G = 0$, $D = -H^2$. Use the abbreviations $S = E + 2HJ$, $\beta = F + J^2$. Then (E), (F), (C) give

$$Lj = c - S, \qquad m = \beta + Lk, \qquad Lh = -C - 2cJ - k(\beta + Lk).$$

By (B), $c = \lambda/\mu$, $\lambda = -BL + S(\beta + Lk)$, $\mu = 2HL + \beta + Lk$. Then (A) becomes a quintic in k, the coefficient of k^5 being $-L^4$.

Third, if $n = L = 0$, (F) and (E) give m and c. If $m \neq 0$, (C), (B), (A) give k, j, h, so that there is an unique rational solution.

8. There remains the case $b = g = 0$, $a \neq 0$. Multiplying x by $1/a$, and y by a^2, we may set $a = 1$. Hence we examine conditions (14) when $b = g = 0$, $a = 1$.

First, let $J \neq 0$. Replacing w by $w - zH/(2J)$, we have $H = 0$. Then equations (14), other than the first two, give

$$L = 0, \qquad p = -J, \qquad n = -e, \qquad d = -G, \qquad k = (e^2 + F)/J,$$

$$c = je - G^2 - D, \qquad m = Jj + 2Ge + ke - E, \qquad h = (C + 2ce + km)/J.$$

We retain the abbreviations k and $\rho = 2Ge - E + ke$, whence $m = Jj + \rho$. Equations (B) and (A) of (14) become

$$Jj^2 - Ej + \alpha = 0, \quad \alpha \equiv B + 2G(G^2 + D) - (Ce + 2ce^2)/J - ke\rho/J,$$

$$(2e^2 + F)j^2 + \beta j + \gamma = 0, \quad \beta \equiv 2\rho k + C + 2ce - 2e(G^2 + D),$$

$$\gamma \equiv A + (G^2 + D)^2 + (C + 2ce)\rho/J + k\rho^2/J.$$

If in each coefficient of these two quadratics in j we retain only the term of maximum degree in e, we find that their resultant becomes $5e^{16}/J^6$. Hence e is a root of an equation of degree 16.

Second, let $J = 0$. Then equations (14), other than the first three, give

$$L = 0, \qquad p = 0, \qquad n = -e - H, \qquad d = -G, \qquad e^2 = -F,$$

$$c = j(e + H) - G^2 - D, \qquad m = k(e + H) + 2Ge - E.$$

The possibility of a (unique) solution with $e = -H$ is easily decided since equations (A), (B), (C) determine uniquely h, j, k if $m \neq 0$. When $e \neq -H$, $e \neq 0$, we express the unknowns in terms of k and e, retaining the abbreviation m. By (C), $c = (-C - km)/(2e)$. Equating this to the above value of c, we get j. Then (B) gives h. Hence (A) becomes

$$4Ae^2(e + H)^2 + (C + km)^2(e + H)^2 + 4Bme^2(e + H)$$
$$+ 4(C + km)Gme(e + H) + 2m^2e\{2(G^2 + D)e - C - km\} = 0,$$

which is a quartic in k, the coefficient of k^4 being $(H + e)^3(H - e)$.

9. Consider the case, excluded in § 5, in which l_3, l_6, l_7, l_8 are all free of z and hence are multiples of w. Postponing to the end of § 11 the case in which (11) is the product of y by its minor, we may assume, in view of the

substitutions S and T of § 3, that l_7 is the product of w by a constant, not zero, which may be removed as a factor from the last row and multiplied into the last column of (2). Thus $l_7 = w$. We may take $l_8 = 0$ by (4). Then $-l_3 w = az^2 + bw^2$ by (3). Thus $a = 0$ and hence $b = 0$ by § 5. Then $l_3 \equiv 0$. Then $l_6 \not\equiv 0$ and by removing its coefficient from the second row and multiplying it into the second column, we may set $l_6 = w$. Hence

$$f = \begin{vmatrix} x + l_1 & l_2 & 0 \\ l_4 & x - l_1 & w \\ w & 0 & y \end{vmatrix} = x^2 y - y(l_1^2 + l_2 l_4) + w^2 l_2.$$

For $y = 0$, f reduces to $w^2\lambda$, if $l_2 = hy + \lambda$. Hence in

$$f_3 = -yQ + K, \qquad Q = Ay^2 + 2Byz + 2Cyw + Dz^2 + 2Ezw + Fw^2,$$

the cubic function K of z, w must have the factor w^2, the quotient being $\lambda = jz + kw$. Further, Q must be of the form $l_1^2 + l_2 l_4 - kw^2$. We shall examine the last question independently of our main problem. If $j \neq 0$, we introduce the known function λ as a new variable z; let Q become $Q' = A'y^2 + \cdots$. By (5) we may subtract a multiple of $l_2 = hy + z$ from l_1 and assume that $l_1 = cy + ew$ lacks z. When Q' is divided by l_2 to give a remainder free of z, the quotient gives l_4 and the remainder must be $l_1^2 - hw^2$. Hence the latter must equal the value of Q' for $z = -hy$, the conditions for which are

$$c^2 = A' - 2B'h + D'h^2, \qquad ce = C' - E'h, \qquad e^2 = h + F'.$$

Hence the cubic

$$(A' - 2B'h + D'h^2)(h + F') = (C' - E'h)^2$$

must have a rational root h such that $h + F'$ is a rational square e^2.

Next, let $j = 0$. Then $k \neq 0$ since we assume that y is not a factor of f. We may take $l_1 = cy + dz$ free of w by (15). When Q and $l_1^2 - hw^2$ are divided by l_2, the remainders must be equal and a comparison of the quotients determines l_4. Thus the function obtained from Q by replacing w by $-yh/k$ must be identical with $(cy + dz)^2 - y^2h^3/k^2$, whence

$$d^2 = D, \qquad cd = B - \frac{Eh}{k}, \qquad c^2 = A - 2\frac{Ch}{k} + F\frac{h^2}{k^2} + \frac{h^3}{k^2}.$$

10. Consider the case $v = 0$, excluded in § 5, in which $l_8 = 0$, $l_7 = z + uw$. By (3), $l_3 = -a(z - uw)$, $b = -au^2$.

If $a = b = 0$, determinant (2) now equals

$$x^2 y - y(l_1^2 + l_2 l_4) + l_2 l_6 l_7. \tag{36}$$

This shall equal $f \equiv x^2y - yQ + K$, for Q and K as in § 9. Thus $K = \lambda l_6 l_7$. We postpone to § 11 the simple case $K \equiv 0$. One of the factors of K determines $l_7 = z + uw$. The other two factors determine λ and l_6 up to constant factors; but our determinant (36) is not altered when l_6 and l_4 are multiplied by $t \neq 0$ and $l_2 = hy + \lambda$ is multiplied by $1/t$. Hence we may regard λ, l_6 and l_7 as fully determined. We introduce λ as a new variable z and proceed as in § 9.

Next, let $a \neq 0$. In view of (5) we may take $l_6 = rw$. If $r \neq 0$, we apply the substitution $(l_2l_4)(l_3l_7)(l_6l_8)$, which corresponds to the interchange of rows and columns, and have the case $v \neq 0$ treated in § 5. If $r = 0$, the determinant is identical with (1) if

$$-y(l_3^2 + l_2l_4) - al_1(z + uw)(z - uw) \equiv f_3,$$

a condition which is exactly of the type last discussed.

11. Consider factorable cubic forms yQ, hitherto excluded. If Q itself has the factor y, yQ equals a determinant whose elements outside the diagonal are all zero. In the contrary case, we may take $Q = cx^2 + \cdots$, where $c \neq 0$, after applying a linear transformation on x, z, w with rational coefficients. Replacing cy by a new y, we have yQ, the coefficient of x^2 in Q being unity. Making a suitable addition to x, we obtain $Q = x^2 + q$, where q is a quadratic form in y, z, w. Thus yQ is of the form (1) with $f_2 \equiv 0$, $f_3 \equiv yq$.

First, consider equations (14) when $a = b = G = H = J = L = 0$, $g \neq 0$. The last seven give at once

$$p = j = 0,\quad k = 2d,\quad n = -2e,\quad gm = F + e^2,\quad gh = -D - d^2,\quad 2gc = E - 2de.$$

Multiplying the first equation (14) by $-g^2$ and the next two by g, we get

$$dE + 2eD + gB = 0,\quad 2dF + eE + gC = 0,\quad d^2F + e^2D + deE = Ag^2 + \tfrac{1}{4}\Delta,$$

where $\Delta = E^2 - 4DF$. Multiply them by e, d, -2, respectively, and add. We get

$$\begin{aligned} 2g^2A + \ \ geB + \ \ gdC &= -\tfrac{1}{2}\Delta, \\ g^2B + 2geD + \ \ gdE &= 0, \\ g^2C + \ \ geE + 2gdF &= 0, \end{aligned}$$

the last two being our first two equations multiplied by g. The determinant of the coefficients of g^2, ge, gd is 8 times the determinant δ of

$$q = Ay^2 + Byz + Cyw + Dz^2 + Ezw + Fw^2. \tag{37}$$

Hence $8\delta g^2 = -\frac{1}{2}\Delta(-\Delta)$. Thus if $\delta \neq 0$, $\Delta \neq 0$, the equations have solutions g, e, d with $g \neq 0$, and g will be rational if δ is a square. If $\Delta = 0$,

the equations have no solutions with $g \neq 0$ if any two-rowed minor is not zero. But if those minors are all zero, there is a solution with $g \neq 0$ except when $q = Ay^2$, $A \neq 0$. Hence the cubic $y(x^2 + q)$ has a rational determinantal representation of the type in § 5 only when the determinant of q is a rational square $\neq 0$ and its minor $E^2 - 4DF$ is not zero, or in the trivial case when q is a perfect square. If we make the substitution $z = Z + eY$, $w = W + dY$, $y = gY$, we see that q becomes $DZ^2 + EZW + FW^2 - \frac{1}{4}\Delta Y^2$ and that our determinant of type (11) reduces to the product of g by (38), which is the value of the initial determinant when $d = e = B = C = 0$, $g = 1$.

Milder restrictions are imposed by the method of § 10 for $a = b = 0$, $K \equiv 0$. Thus $\lambda l_6 l_7 \equiv 0$. If $l_6 \equiv 0$, the determinant equals the product of y by its minor, a case treated below. Hence $\lambda \equiv 0$, $l_2 = hy$. The case $h = 0$ is of the type just postponed. Writing hl_4 as a new l_4 in (36), we may take $h = 1$. In view of (5) we may assume that l_1 is free of y. Our cubic is of the form (36) if $q \equiv - l_1^2 - yl_4 + l_6 l_7$. Hence, by (37), $- l_4 = Ay + Bz + Cw$. It remains to choose

$$l_1 = dz + ew, \qquad l_6 = rz + sw, \qquad l_7 = vz + uw$$

so that $Dz^2 + Ezw + Fw^2 \equiv - l_1^2 + l_6 l_7$. The conditions are

$$vr = d^2 + D, \qquad ur + vs = 2de + E, \qquad us = e^2 + F.$$

These are linear equations in $r, s, -1$, the determinant of whose coefficients is

$$u^2(d^2 + D) - uv(2de + E) + v^2(e^2 + F) = 0,$$

$$- u^2 D + uvE - v^2 F = (ud - ve)^2.$$

The problem is solvable if we can choose rational numbers u, v, not both zero, such that the left member* is the square ρ^2 of a rational number. Then rational values of d, e may be chosen so that $ud - ve = \rho$. Hence $y(x^2 + q)$ has a rational determinantal representation of the present type if and only if $x^2 + q$ vanishes at a rational point having $y = 0$.

If our cubic is of the form in § 9, we must have $\lambda \equiv 0$, and

$$q = - l_1^2 - hyl_4 + hw^2,$$

where we may assume that $h \neq 0$ and that l_1 is free of y (as in the preceding case). Thus $- hyl_4$ equals the sum of the first three terms of (37) and $- l_1^2 + hw^2$ equals the sum of the last three. The conditions on $l_1 = dz + ew$ are $d^2 = - D$, $2de = - E$, $h = e^2 + F$. These determine d, e, h rationally if $- D$ is a rational square $\neq 0$, and are satisfied if $D = E = 0$. In the

* That this condition is necessary is seen by taking $z = u$, $w = - v$ in the proposed identity, whence $l_7 = 0$.

respective cases, $x^2 + q$ vanishes at a rational point having $y = w = 0$ or $x = y = w = 0$.

It remains to consider the frequently postponed case of the rational representation of a product of a linear form l and a quadratic form Q as a determinant in which the elements of a row or column are $0, 0, l$. Thus Q is to be represented as a two-rowed determinant. An evident necessary condition is that Q vanish at a rational point. Then, as in § 2, Q can be transformed rationally into $xy + q(z, w)$ or $q(y, z, w)$. Any two-rowed determinant equal to the former may be given the form

$$\begin{vmatrix} x + A & B \\ C & y \end{vmatrix},$$

where B and C are free of x and y, whence $A \equiv 0$, $q = -BC$. Similarly, if $q(y, z, w)$ equals a rational determinant, it vanishes for rational values not all zero of y, z, w, and is rationally equivalent to $yz + kw^2$ or to a binary form. In the following summary, the various cases are presented in reverse order.

THEOREM. *A product of a linear form l and quadratic form Q in four variables with rational coefficients can be expressed as a determinant whose elements are linear functions with rational coefficients only in the following cases:* (i) *Q is expressible rationally as a two-rowed determinant if it represents a cone which vanishes at a rational point not the vertex, or if Q vanishes at a rational point P and the tangent plane at P cuts the surface in rational lines.* (ii) *Q and l both vanish at the same rational point.* (iii) *lQ is rationally equivalent to $y(x^2 + q)$, where q is the ternary form* (37) *whose determinant is the square of a rational number $\neq 0$ and its minor $\Delta = E^2 - 4DF$ is not zero; the resulting representation is derived from the special case*

$$(38)\quad Y(x^2 + DZ^2 + EZW + FW^2 - \tfrac{1}{4}\Delta Y^2) = \begin{vmatrix} x + \frac{1}{2}EY & -DY & W \\ FY & x - \frac{1}{2}EY & -Z \\ Z & W & Y \end{vmatrix}$$

by a linear substitution of type $z = Z + eY$, $w = W + \alpha Y$, $y = gY$.

COROLLARY. If A, D, F are rational numbers for which

$$Q = x^2 + AY^2 + DZ^2 + FW^2$$

vanishes at no rational point, YQ is representable rationally as a determinant if and only if ADF is a rational square.

I have found a simple proof of this Corollary independently of the present theory.

12. In view of §§ 5, 9, 10, $x^2y + Gz^3 + LW^3$ is expressible rationally in determinantal form only when $Gz^3 + LW^3$ is a product of three factors with rational coefficients whence $GL = 0$. But if K is any homogeneous cubic, $x^2y - K(x, z, w) = 0$ evidently has the solutions

$$x = \rho A^3, \qquad y = \rho K(A, B, C), \qquad z = \rho A^2 B, \qquad w = \rho A^2 C,$$

and no further rational solutions when $x \neq 0$. For, if A is any rational number $\neq 0$, we may define ρ by $x = \rho A^3$, B by $z = \rho A^2 B$, C by $w = \rho A^2 C$; then y has the value specified.

These solutions do not imply a determinantal representation of the surface since there do not exist three linearly independent linear relations between $x, \cdots, w$ with coefficients linear in A, B, C, provided K does not have the factor x. In fact, all such relations are linear combinations of $Bx - Az = 0$, $Cx - Aw = 0$. But if $K \equiv y^2 z$, for example, we may secure such relations by taking new parameters A, B, $D = A^2/C$; then, for $\rho = \sigma D/A^2$,

$$x = \sigma AD, \qquad y = \sigma B^2, \qquad z = \sigma BD, \qquad w = \sigma A^2,$$

$$Bx - Az = 0, \qquad -Ax + Dw = 0, \qquad -Bz + Dy = 0.$$

13. The question whether the special cases postponed in § 5 are really exceptional is best answered by a different approach to the problem of normalizing our initial determinant. Its matrix is $M = xA + yB + zC + wD$, where, by (2),

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} t_1 & t_2 & 0 \\ t_4 & -t_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

t_i being the coefficient of y in l_i. To further normalize M, we have available any constant matrices P and Q, whose determinants are not zero, such that in PMQ the coefficient of x is our A, while that of y is a matrix B' of type B with t_i replaced by t'_i. Thus $PAQ = A$, $PBQ = B'$. Set $R = P^{-1}$. Then $AQ = RA$, $BQ = RB'$, which are easily seen to require that

$$R = Q = \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \rho \end{bmatrix}.$$

Thus $PMQ = R^{-1}MR$, so that the normalization of M must arise by transformation by a matrix R of the form just given. Let c_{ij} and d_{ij} denote the elements of C and D in the ith row and jth column. By (2), $c_{33} = d_{33} = 0$, $c_{22} = -c_{11}$, $d_{22} = -d_{11}$.

The first case postponed in § 5 is that in which c_{13}, c_{23}, c_{31}, and c_{32} are all zero. Since $R^{-1}CR$ then has the same four elements zero, this case is truly exceptional. Excluding it, we may take* $c_{31} \neq 0$. By choice of ρ,

* For $\alpha = \delta = 0$, $\beta = \gamma = \rho = 1$, transformation of C by R interchanges c_{13} and c_{23}, c_{31} and c_{32}. We also allow passing to the transposed matrix (with rows and columns interchanged), thus treating one of two similar problems. Note that the transposed matrix cannot be obtained from C by transforming by a matrix of the specal form R.

we may take $c_{31} = 1$. Transforming by R with $\alpha = \delta = \rho = 1$, $\gamma = 0$, $\beta = -c_{32}$, we may now have $c_{32} = 0$ as well as $c_{31} = 1$, $c_{33} = 0$. If also the last row of $R^{-1}CR$ is 1, 0, 0, we must have $\beta = 0$, $\alpha = \rho$. The last row of $R^{-1}DR$ is then

$$d_{31} + \frac{\gamma}{\rho} d_{32}, \qquad \frac{\delta}{\rho} d_{32}, \qquad 0.$$

Hence the special case in which d_{32} (denoted by v in § 5) is zero is truly exceptional. Excluding it, we may choose γ/ρ and δ/ρ so that $d_{31} = 0$, $d_{32} = 1$. Then if R is such that the last row of $R^{-1}DR$ is still 0, 1, 0, we have $\gamma = 0$, $\delta = \rho$, so that R is a similarity-matrix having the diagonal elements equal and having the remaining elements all zero. Thus R transforms every matrix into itself and no further normalization of B, C, D is possible.

14. The difficult part of our problem is to identify the final determinant in (11) with any given ternary form (13) in y, z, w. The matrix of that determinant is $M = yY + zZ + wW$, where

$$Y = \begin{bmatrix} c & h & 0 \\ m & -c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Z = \begin{bmatrix} d & j & -a \\ n & -d & -g \\ 1 & 0 & 0 \end{bmatrix}, \qquad W = \begin{bmatrix} e & k & g \\ p & -e & -b \\ 0 & 1 & 0 \end{bmatrix}.$$

Let Y', Z', W' denote the similar matrices in $c', \cdots, g'$, but with the same a and same b. Do there exist matrices P and Q with constant elements of determinants not zero such that $PMQ = M'$? If so, the determinants of M and M' differ only by a constant factor which we may assume is unity. It is more convenient to treat $MQ = RM'$, where $R = P^{-1}$. There are really several questions, depending upon what is assumed to be given.

First, let only the determinant of M be given and require all matrices $M, M', \cdots$ of our special form and investigate their equivalence. Since this is our initial difficult problem with a supplement, we will illustrate the facts by means of an instructive example. Let $|M| = w^3 - z^3 - yzw$, so that $E = G = -1$, $L = 1$, while the remaining coefficients of (13) are zero.* Thus (15)–(17) become

$$-3\epsilon^2 + \delta(3 - g^2) = 0, \quad 3\delta^2 + \epsilon(3 - g^2) = 0, \quad \delta\epsilon(21 - g^2) + \tfrac{3}{4}(1 - g^2)^2 = 0.$$

By the first two, $\delta\epsilon = 0$ or $-(3 - g^2)^2/9$. If $\delta\epsilon = 0$, then $g = \pm 1$, $\delta = \epsilon = 0$. If $\delta\epsilon \neq 0$, $(g^2 - 9)^2(g^2 - 9/4) = 0$. If $g = \pm 3$, $\delta = -2$, $\epsilon = 2$. If $g = \pm 3/2$, $\delta = 1/4$, $\epsilon = -1/4$. The equations preceding (15) become

$$gp = gj = 1, \qquad gn = -2\epsilon, \qquad gk = 2\delta, \qquad g^3 m = \epsilon^2 + 2\delta,$$
$$g^3 h = 2\epsilon - \delta^2, \qquad 2g^3 c = 1 - g^2 - 2\delta\epsilon.$$

* The only singular point of $x^2 y + t_3 = 0$ is (0, 1, 0, 0).

Changing the sign of g merely changes the signs of the elements of the first two rows of M. Hence we take $g = 1, 3, 3/2$ in turn and obtain

$$Y = Y' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

$$Z' = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{4}{3} & \frac{2}{3} & -3 \\ 1 & 0 & 0 \end{bmatrix}, \quad W' = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3} & 3 \\ \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$M'' = \begin{bmatrix} \frac{1}{6}(-y+z-w), & \frac{1}{6}(-y+4z+2w), & \frac{3}{2}w \\ \frac{1}{6}(y+2z+4w), & \frac{1}{6}(y-z+w), & -\frac{3}{2}z \\ z & w & y \end{bmatrix}.$$

The conditions $YQ = RY'$, $ZQ = RZ'$, $WQ = RW'$ are satisfied if and only if

$$R = r\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}, \quad Q = r\begin{bmatrix} -1 & 2 & -6 \\ 2 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix},$$

so that M and M' are equivalent. But in M'' the coefficient Y'' of y is of rank 2, while Y is of rank 1, so that M and M'' are not equivalent. Hence two matrices of type M with the same determinant may or may not be equivalent.

A second question relates to a possible simplification of our initial problem in advance of its solution. Let a and b be given, while the remaining parameters $c, h, \cdots, g$ in M are indeterminates. Can we find matrices R and Q with elements independent of $c, \cdots, g$ such that $MQ = RM'$ for suitably determined elements $c', \cdots, g', a, b$ of M'? If we can find such matrices R and Q not both similarity matrices rI and qI, we can employ them to normalize M formally in advance of its computation. Denote the elements of the ith row and jth column of R and Q by r_{ij} and q_{ij} respectively. By the third columns of $YQ = RY'$,

$$cq_{13} + hq_{23} \equiv r_{13}, \qquad mq_{13} - cq_{23} \equiv r_{23},$$

identically in c, h, m. Hence $q_{13} = q_{23} = r_{13} = r_{23} = 0$. By the third elements of the third rows of $ZQ = RZ'$ and $WQ = RW'$,

$$-ar_{31} - g'r_{32} = q_{13} = 0, \qquad g'r_{31} - br_{32} = q_{23} = 0.$$

By hypothesis, their determinant $ab + g'^2$ is not zero. Hence $r_{31} = r_{32} = 0$. Then by the third rows of our matrix products,

$$q_{11} = q_{22} = q_{33} = r_{33}, \qquad q_{ij} = 0 \quad (i \neq j), \qquad Q = r_{33}I.$$

Since we may replace R by $r_{33}^{-1}R$, we may take $r_{33} = 1$. We now employ matrix $P = R^{-1} = (p_{ij})$, which has $p_{13} = p_{23} = 0, p_{33} = 1$. Then $PM = M'$. In

$$PY = \begin{bmatrix} cp_{11} + mp_{12} & hp_{11} - cp_{21} & 0 \\ cp_{21} + mp_{22} & hp_{21} - cp_{22} & 0 \\ cp_{31} + mp_{32} & hp_{31} - cp_{32} & 1 \end{bmatrix} = Y',$$

we see by the last row and the sum of the diagonal elements that

$$p_{31} = p_{32} = p_{11} - p_{22} = p_{12} = p_{21} = 0.$$

Then $p_{11}^2 = 1$ by $|M| = |M'|$. For $p_{11} = +1$, we have the trivial case $P = Q = I$. For $p_{11} = -1$, the multiplication of M by P on the left is equivalent to changing the signs of the elements of the first two rows. But the elements a and b in Z and W are to remain unchanged. Hence *formal normalization of M is possible only when $a = b = 0$ and then consists in changing the signs of the elements of the first two rows.* Hence the equation for g then involves only even powers.

THE IMPOSSIBILITY OF EINSTEIN FIELDS IMMERSED IN FLAT SPACE OF FIVE DIMENSIONS.

By Edward Kasner.

By the theory of quadratic differential forms it follows that a general riemannien manifold of m dimensions can always be regarded as immersed in some flat space of n dimensions, where n does not exceed $\frac{1}{2}m(m+1)$. Thus if $m = 4$, as in the Einstein theory, the form

$$ds^2 = \Sigma g_{ik} dx_i dx_k \qquad (i, k = 1, 2, 3, 4)$$

can be immersed in an n-flat, where the possible values of n are 4, 5, 6, 7, 8, 9, 10.

If now we require the manifold to obey Einstein's equations of gravitation, $G_{ik} = 0$, the question arises which of these values of n are actually realizable. The case $n = 4$ is trivial, since then the curvature vanishes and there is no permanent gravitation. *We wish to show now that the case $n = 5$ is impossible; that is, no Einstein manifold can be regarded as imbedded in a five-flat.*

In a flat space of five dimensions let rectangular coördinates be denoted by x_1, x_2, x_3, x_4, w. Then any four-dimensional manifold M in that space may be defined by a single finite equation

(1) $$w = f(x_1, x_2, x_3, x_4).$$

The element of length of M is

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dw^2,$$

where

$$dw = f_i dx_i, \qquad \left(f_i = \frac{\partial f}{\partial x_i}\right),$$

is the total differential, summation with respect to the repeated index i being understood as usual. This can be written

(2) $$ds^2 = g_{ik} dx_i dx_k,$$

where

(3) $$g_{11} = 1 + f_1^2, \qquad g_{12} = f_1 f_2, \qquad \text{etc.}$$

The determinant of (2) is

(4′) $$g = \begin{vmatrix} 1 + f_1^2 & f_1 f_2 & f_1 f_3 & f_1 f_4 \\ f_1 f_2 & 1 + f_2^2 & f_2 f_3 & f_2 f_4 \\ f_1 f_3 & f_2 f_3 & 1 + f_3^2 & f_3 f_4 \\ f_1 f_4 & f_2 f_4 & f_3 f_4 & 1 + f_4^2 \end{vmatrix},$$

which reduces exactly to

(4) $$g = 1 + f_1^2 + f_2^2 + f_3^2 + f_4^2.$$

The minors of (4′) also simplify, giving

(5) $$g^{11} = \frac{1 + f_2^2 + f_3^2 + f_4^2}{g}, \qquad g^{12} = \frac{f_1 f_2}{g}, \quad \text{etc.}$$

Our problem is to find the function f so that the ten functions (3) shall satisfy the ten gravitational equations* $G_{ik} = 0$.

CALCULATION OF THE CHRISTOFFEL SYMBOLS.

We now calculate the Christoffel symbols (of the second kind)

(6) $$\{\alpha\beta, \gamma\} = \tfrac{1}{2} g^{\gamma\epsilon}(g_{\alpha\epsilon,\beta} + g_{\beta\epsilon,\alpha} - g_{\alpha\beta,\epsilon}).$$

From (3) we find the derivatives

$$g_{ik,j} = (f_i f_k)_j = f_i f_{kj} + f_k f_{ij},$$

this formula holding whether the subscripts are unequal or equal. Substituting in (6), we find

(6′) $$\{\alpha\beta, \gamma\} = f_{\alpha\beta} f_\epsilon g^{\gamma\epsilon}.$$

Here the second derivative $f_{\alpha\beta}$ appears as a factor, and it remains to carry out the summation $f_\epsilon g^{\gamma\epsilon}$ involving only first derivatives of f. If the variable index ϵ takes the value γ, we have

$$f_\gamma g^{\gamma\gamma} = f_\gamma \frac{1 + \Sigma' f_{\epsilon'} f_{\epsilon'}}{g} \qquad \text{(no summation on left)},$$

where the prime indicates that ϵ' is to vary through the three values different from γ. For the three values of ϵ different from γ, we have

$$\Sigma' f_{\epsilon'} g^{\gamma\epsilon'} = f_\gamma \frac{\Sigma' f_{\epsilon'} f_{\epsilon'}}{g}.$$

Therefore

$$f_\epsilon g^{\gamma\epsilon} = f_\gamma \frac{1 + 2\Sigma' f_{\epsilon'} f_{\epsilon'}}{g}.$$

Hence the final value of our symbol is

(7) $$\{\alpha\beta, \gamma\} = \frac{f_{\alpha\beta} f_\gamma P_\gamma}{g},$$

where

(7′) $$P_\gamma = 1 + 2\Sigma' f_{\epsilon'} f_{\epsilon'} = 2g - 1 - 2f_\gamma^2.$$

* For the notation see Weyl or Eddington, or § 1 of our paper "Einstein's theory of gravitation: determination of the field by light signals," this JOURNAL, vol. 43 (1921), p. 20.

CALCULATION OF THE EINSTEIN TENSOR G_{ik}.

Take first the case of unlike indices,

$$G_{12} = \{1\alpha, \beta\}\{2\beta, \alpha\} - \{12, \alpha\}L_\alpha - \{12, \alpha\}_\alpha + L_{12}, \tag{8}$$

where $L = \frac{1}{2}\log g$. We now simplify the rest of the discussion by taking our axes in the original five-flat so the w-axis is normal to the four-spread (1) and so that the axes x_1, x_2, x_3, x_4 agree with the principal directions of the spread at the selected point, which we select as origin. This means that the partial derivatives of first order f_1, f_2, f_3, f_4 all vanish, and that

$$f_{12} = f_{13} = f_{14} = f_{23} = f_{24} = f_{34} = 0.$$

The four-spread (1) may therefore be written in power series form

$$w = \tfrac{1}{2}(k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + k_4x_4^2) + \text{higher terms.} \tag{9}$$

Thus at the origin

$$f_{11} = k_1, \qquad f_{22} = k_2, \qquad f_{33} = k_3, \qquad f_{44} = k_4,$$

these values defining the principal curvatures. We have also $g = 1$, so that $L = 0$, at the origin.

For this special choice of axes we see from (7) that all the Christoffel symbols vanish. This throws out the first and second terms of (8). The third and fourth terms involve second derivatives of f, but we find that both terms vanish. Hence G_{12}, and of course the other five combinations with unlike subscripts, vanish identically.

Take now the case of like indices

$$G_{11} = \{1\alpha, \beta\}\{1\beta, \alpha\} - \{11, \alpha\}L_\alpha - \{11, \alpha\}_\alpha + L_{11}. \tag{10}$$

Here the first and second terms vanish. The third term is found to be

$$-k_1(k_1 + k_2 + k_3 + k_4).$$

The last term reduces to

$$k_1^2.$$

Hence*

$$G_{11} = -k_1(k_2 + k_3 + k_4). \tag{11}$$

Thus the conditions that the four-spread (9) *shall be of the Einstein type (that is, obey* $G_{ik} = 0$*) are*

---

* From this we may find the known formula for scalar curvature

$$G = g^{ik}G_{ik} = -2(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4).$$

Cf. Eddington, "Report," p. 75, or Page, *Trans. Connecticut Acad.*, Vol. 23 (1920), p. 408. We may also connect with the elegant interpretations of the tensor G_{ik} given by Herglotz, *Leipziger Berichte*, 1916 (not accessible), and Vermeil, *Göttinger Berichte*, 1917.

$$
(12) \qquad
\begin{aligned}
k_1(k_2 + k_3 + k_4) &= 0,\\
k_2(k_3 + k_4 + k_1) &= 0,\\
k_3(k_4 + k_1 + k_2) &= 0,\\
k_4(k_1 + k_2 + k_3) &= 0.
\end{aligned}
$$

On the other hand the conditions that (9) shall be flat (euclidean or homoloidal) are

$$
(13) \qquad k_1k_2 = k_1k_3 = k_1k_4 = k_2k_3 = k_2k_4 = k_3k_4 = 0.
$$

The two sets (12) and (13) are easily seen to be equivalent, each meaning that at least three of the four quantities k_1, k_2, k_3, k_4 must vanish.

This shows that in a flat space of five dimensions we cannot construct four-dimensional manifolds (other than the trivial euclidean type) which obey Einstein's equations. *That is, an Einstein spread representing a permanent gravitational field can never be regarded as immersed in a five-flat.*

In flat space of six dimensions, actual Einstein manifolds exist; in particular, the solar field discussed in the next paper.

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FINITE REPRESENTATION OF THE SOLAR GRAVITATIONAL FIELD IN FLAT SPACE OF SIX DIMENSIONS.

By Edward Kasner.

In the preceding paper it was shown that no solution of the Einstein equations $G_{ik} = 0$ can represent a four-spread imbedded in a five-flat (except in the trivial case where there is no permanent gravitation, so that the four-spread is itself flat). It follows that the solar field can only be imbedded in a flat space of more than five dimensions. By the general theory of quadratic forms in four variables, the maximum number of dimensions required is ten. We shall now show that *the requisite dimensionality for the solar field is actually six.** The finite equations, in six cartesian coordinates, of the curved four-spread representing the solar field are given. As seen in formulas (7) below, a hyperelliptic integral is involved. This spread may be described as a *geometric model of the exact field* in which we are living.

The field may be taken in the usual Schwarzschild form

$$ds^2 = \gamma dt^2 - \gamma^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \tag{1}$$

where

$$\gamma = 1 - \frac{2m}{r};$$

here m is the mass of the sun, r, θ, φ are polar coordinates, and t is the time.

Introducing cartesian coordinates, x, y, z, we have the equivalent form

$$ds^2 = -dx^2 - dy^2 - dz^2 - \frac{2m}{r - 2m} dr^2 + \frac{r - 2m}{r} dt^2, \tag{2}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{2'}$$

In (1) we have four square terms all with variable coefficients. In (2) we have three squares with unit coefficients, and two squares with variable coefficients. Our object is to obtain an equivalent form with six squares, all with unit coefficients. Of course, if we admit imaginary transformations, all the coefficients can be made $+1$; but if we keep to the real domain, a certain number will equal $+1$ and a certain number -1.

The fourth coefficient in (2) is a function of r alone; therefore we introduce a new variable R so that

* Other fields in six dimensions exist and will be discussed elsewhere.

$$dR = \sqrt{\frac{2m}{r - 2m}}\,dr,$$

that is,

(3) $$R = \sqrt{8m(r - 2m)}.$$

Our form then becomes

(4) $$ds^2 = -\,dx^2 - dy^2 - dz^2 - dR^2 + \frac{R^2}{R^2 + 16m^2}\,dt^2.$$

The last term is of course not the square of an exact differential, and our general theorem on five-flats shows that we cannot obtain a form with five unit squares.

Taking the last two terms of (4), which involve merely the two variables R and t, we can, by the usual theory of surfaces, replace their combination

(5) $$dS^2 = -\,dR^2 + \frac{R^2}{R^2 + 16m^2}\,dt^2$$

by a sum of three unit squares. We readily find

(6) $$dS^2 = dX^2 + dY^2 - dZ^2,$$

where

(7) $$X = \frac{R \sin t}{\sqrt{R^2 + 16m^2}}, \qquad Y = \frac{R \cos t}{\sqrt{R^2 + 16m^2}},$$

$$Z = \int \sqrt{1 + \frac{256m^4}{(R^2 + 16m^2)^3}}\,dR.$$

The form (4) may now be written

(8) $$ds^2 = -\,dx^2 - dy^2 - dz^2 + dS^2;$$

but of course it must be remembered that while dx, dy, dz are exact differentials, dS is not. In fact dS is the distance element of the auxiliary surface defined by (7), where X, Y, Z (or rather X, Y, iZ) are cartesian coordinates in an auxiliary three-flat, and R and t are gaussian coordinates on this surface. This transcendental surface (involving a hyperelliptic integral), which apparently has escaped notice, we shall designate by (S). It is obviously a surface of rotation. To really obtain a form with all exact differentials we substitute (6) in (8). This gives our

THEOREM. *The exact solar field may be written with six squares of exact differentials*

(9) $$ds^2 = -\,dx^2 - dy^2 - dz^2 + dX^2 + dY^2 - dZ^2,$$

where X, Y, Z are defined by equations (7) *in conjunction with* (3) *and* (2′), *so that they are known functions of the world coordinates x, y, z, t.*

Thus if we consider a flat space with six rectangular coordinates x, y, z, X, Y, Z (or rather, on account of the minus signs in (9), with coordinates* ix, iy, iz, X, Y, iZ), *the finite representation of the solar field is a curved manifold M_4 of four dimensions (hypersurface) situated in a six flat and defined by* (7) *with* (3) *and* (2′).

If on this M_4 we take the parameter t equal to zero we obtain a sub-manifold M_3 which is peculiarly simple. From (7) we have $X = 0$, therefore the distance element of M_3 is, from (9),

$$-dx^2 - dy^2 - dz^2 + dY^2 - dZ^2, \tag{10}$$

where

$$X = \frac{R}{\sqrt{R^2 + 16m^2}}, \qquad Z = \int \sqrt{1 + \frac{256m^4}{(R^2 + 16m^2)^3}}\, dR. \tag{10′}$$

This shows that M_3 is in the five-flat (x, y, z, Y, Z). But going back to the form (4), and substituting $dt = 0$, we have as an equivalent element of M_3,

$$-dx^2 - dy^2 - dz^2 - dR^2; \tag{11}$$

or, changing all the signs, we use as the final form

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dR^2, \tag{12}$$

where R is defined by (3). This is obviously an M_3 in a four-flat. It is easily seen to be a rotation manifold. In particular if we put $z = 0$, we have

$$dx^2 + dy^2 + dR^2, \tag{13}$$

which defines a paraboloid obtained by rotating the parabola

$$R = \sqrt{2m(r - 2m)}, \tag{13′}$$

supposed drawn in the (r, R) plane, about its directrix (the axis of R). This simple result, defining the geometry in a plane through the sun, was first obtained by FLAMM.† It illustrates *the spatial measurements* for a given value of the time parameter t.

The surface (S) defined by (6) and (7) is obviously entirely different, and illustrates *the relation of the space and time measurements.* It is also a

* In Weyl's terminology this would be described by saying that the flat space here employed has four negative and two positive dimensions. The six space is not properly euclidean, but what Hilbert (*Göttinger Nachrichten*, 1915, 1917) would term pseudo-euclidean.

† L. Flamm, "Beiträge zur Einsteinschen Gravitationstheorie," *Physik. Zeitschr.*, Vol. 25 (1916), p. 163. A simple exposition is given in Weyl, "Raum, Zeit, Materie," 3d edition (1919), p. 223.

surface of rotation, but the generating curve is transcendental. From (7), we find

$$Z = \frac{1}{2} \int \sqrt{\frac{(1 - P)^3 + 16m^2}{P(1 - P)^3}}\, dP, \tag{14}$$

where

$$P = p^2 = \frac{R^2}{R^2 + 16m^2} = \frac{r - 2m}{r}. \tag{14'}$$

If we take p as abscissa and Z as ordinate we have the required curve. Rotation about the axis of ordinates (with Z replacement by iZ) produces the surface (S). The time parameter t is represented by the angle of rotation.

Flamm's paraboloid, defined by (13′), *and the new surface, defined by* (14), *thus supplement each other, and together characterize the metric of the solar field.*

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ON THE MOTION OF TWO SPHEROIDS IN AN INFINITE LIQUID ALONG THEIR COMMON AXIS OF REVOLUTION.

BY BIBHUTIBHUSAN DATTA.

INTRODUCTION.

Though the problem of the motion of two spheres in an infinite liquid along the line joining their centers has been completely solved by various investigators, the first writer to attempt the corresponding problem for two spheroids or ellipsoids is Prof. Karl Pearson.* His method does not, however, admit of further development and therefore, does not lead to the complete solution of the problem. In a previous paper,† I have shown how the problem can be completely solved in the case of two spheroids of revolution of *small* ellipticities, the motion of the solids being along their common axis of revolution. The present paper deals with the more *general* case of the same problem in as much as no limitation has been imposed as regards the ellipticities. It will be seen that the success of the problem depends on certain transformation theorems for spheroidal harmonics which were not known before, though the corresponding theorems for spherical harmonics were given long ago by Bessel.

All the results in this paper are believed to be new.

§ 1. LEMMA.

1. Let r_1, θ_1, ω_1 and r_2, θ_2, ω_2 be two systems of spherical polar coördinates having their origins at O_1 and O_2 respectively and their polar axes in the directions of O_1O_2 and O_2O_1; also let x_1, y_1, z_1 and x_2, y_2, z_2 be two parallel systems of rectangular cartesian coördinates with the same origins, the polar axes being taken for the axes of x_1 and x_2; let s denote the distance between O_1 and O_2.

Since

$$\rho_2 = \sqrt{y_2^2 + z_2^2} = \rho_1$$

we have

* "On the Motion of Spherical and Ellipsoidal Bodies in Fluid Media," Part II (*Quarterly Journal of Pure and Applied Mathematics*, Vol. XX).

† *Bulletin of the Calcutta Mathematical Society*, Vol. VII.

$$\frac{P_n(\cos\theta_2)}{r_2^{n+1}} = \frac{1}{\pi}\int_0^\pi \frac{d\vartheta}{(x_2 + i\rho_2\cos\vartheta)^{n+1}}, \qquad i = \sqrt{-1}$$

$$= \frac{1}{\pi}\int_0^\pi \frac{d\vartheta}{\{s - (x_1 + i\rho_1\cos\vartheta)\}^{n+1}};$$

when $|s| > |x_1 + i\rho_1\cos\vartheta|$, we can expand the right-hand side of this equation in a series of spherical harmonics,* so that

$$\frac{P_n(\cos\theta_2)}{r_2^{n+1}} = \frac{(-1)^n}{n!}\frac{1}{k_1^{n+1}}\cdot\frac{1}{\pi}\int_0^\pi \left[\sum_{m=0}^{\infty}(2m+1)P_m\left(\frac{x_1+i\rho_1\cos\vartheta}{k_1}\right)\frac{d^nQ_m(t)}{dt^n}\right]d\vartheta,$$

where $t = s/k_1$ and k_1 is any quantity.

Following the notation of Sir W. D. Niven in his memoir "On Ellipsoidal Harmonics" (*Phil. Trans.*, Vol. 128 (1891), *A*), let $\mathfrak{S}(2)$ denote the external harmonics of a system of confocal spheroids having the center at O_2 and O_1O_2 as the axis of revolution. Then $\mathfrak{S}(2)$ can be expressed in a series of spherical harmonics in the following way:†

$$\mathfrak{S}(2) = (-1)^n\frac{2^{n+1}n!}{(2n+1)!}H_n\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right)\left\{1 + \frac{D^2}{2(2n+3)} + \frac{D^4}{2\cdot4\cdot(2n+3)(2n+5)} + \cdots\right\}\frac{1}{\sqrt{x_2^2+y_2^2+z_2^2}},$$

where D^2 stands for $(a_2^2 - c_2^2)(\partial^2/\partial x_2^2)$, and a_2, c_2 $(a_2 > c_2)$ are the semi-axes of any one of the quadrics of the second family.

Now

$$H_n\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right)\frac{1}{r_2} = (-1)^n\frac{(2n)!}{2^n n!}\frac{H_n(x_2, y_2, z_2)}{r_2^{2n+1}},‡$$

and

$$H_n(x_2, y_2, z_2) = r_2^nP_n(\cos\theta_2).$$

Therefore

$$H_n\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right)\frac{1}{r_2} = (-1)^n\frac{(2n)!}{2^n n!}\frac{P_n(\cos\theta_2)}{r_2^n}$$

$$= (-1)^n\frac{(2n)!}{2^n n!}\frac{1}{\pi}\int_0^\pi\frac{d\vartheta}{(x_2 + i\rho_2\cos\vartheta)^{n+1}}.$$

Hence if k_2^2 denotes $a_2^2 - c_2^2$ we get

* Vide Todhunter, "The Functions of Laplace, Lame and Bessel," p. 88.

† Vide Niven's memoir, p. 245; Hobson, "On the Evaluation of a Certain Surface Integral and its Application to the Expansion in Series of the Potential of Elipsoids," *Proc. Lond. Math. Soc.*, Vol. XXIV, p. 91.

‡ Niven's memoir, p. 236.

$$\mathfrak{S}(2) = \frac{2}{(2n+1)}\frac{1}{\pi}\int_0^\pi \left[\frac{1}{(x_2 + i\rho_2 \cos\vartheta)^{n+1}} + \frac{(n+1)(n+2)}{2(2n+3)}\frac{k_2^2}{(x_2 + i\rho_2\cos\vartheta)^{n+3}} + \cdots\right] d\vartheta$$

$$= \frac{(2n)!}{2^{n-1}n!\,n!}\frac{1}{k_2^{n+1}}\frac{1}{\pi}\int_0^\pi Q_n\left(\frac{x_2 + i\rho_2\cos\vartheta}{k_2}\right) d\vartheta.$$

Again

$$\mathfrak{S}(2) = \frac{2}{(2n+1)}\frac{(-1)^n}{n!}\frac{1}{k_1^{n+1}}\frac{1}{\pi}\int_0^\pi \sum_{m=0}^{\infty}(2m+1)P_m\left(\frac{x_1 + i\rho_1\cos\vartheta}{k_1}\right) d\vartheta \times \left[\frac{d^n}{dt^n} + \frac{(k_2/k_1)^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}} + \cdots\right] Q_m(t).$$

Therefore we have

$$\frac{(2n+1)!}{2^n n!\,n!}\frac{1}{k_2^{n+1}}\frac{1}{\pi}\int_0^\pi Q_n\left(\frac{x_2 + i\rho_2\cos\vartheta}{k_2}\right) d\vartheta$$

$$= \frac{(-1)^n}{n!}\frac{1}{k_1^{n+1}}\frac{1}{\pi}\int_0^\pi \sum_{m=0}^{\infty}(2m+1)P_m\left(\frac{x_1 + i\rho_1\cos\vartheta}{k_1}\right) d\vartheta \times \left[\frac{d^n}{dt^n} + \frac{\sigma_1^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}} + \cdots\right] Q_m(t), \quad \text{(I)}$$

where σ_1 stands for k_2/k_1.

We shall now express this in terms of spheroidal coördinates. Supposing the spheroids are *prolate*, we write

$$x_1 = k_1 \cos\theta_1 \cosh\eta_1 = k_1\mu_1\lambda_1,$$

$$\rho_1 = k_1 \sin\theta_1 \sinh\eta_1 = k_1(1-\mu_1^2)^{1/2}(\lambda_1^2 - 1)^{1/2};$$

then

$$\frac{1}{\pi}\int_0^\pi P_m\left(\frac{x_1 + i\rho_1\cos\vartheta}{k_1}\right) d\vartheta = P_m(\mu_1)P_m(\lambda_1),$$

and

$$\frac{1}{\pi}\int_0^\pi Q_n\left(\frac{x_2 + i\rho_2\cos\vartheta}{k_2}\right) d\vartheta = P_n(\mu_2)Q_n(\lambda_2).$$

Substituting in (I) we get

$$P_n(\mu_2)Q_n(\lambda_2) = \sum_{m=0}^{\infty}(2m+1)\hat{\omega}_1(m,n)P_m(\mu_1)P_m(\lambda_1), \quad (A)$$

where

$$\hat{\omega}_1(m,n) = (-1)^n\frac{2^n n!}{(2n+1)!}\sigma_1^{n+1}\left[\frac{d^n}{dt^n} + \frac{\sigma_1^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}} + \cdots\right] Q_m(t).$$

If however the spheroids are of the *oblate or planetary* form, $a_1^2 - c_1^2$ is negative, so that we write $\gamma_1^2 = c_1^2 - a_1^2$, and the corresponding coördinates are

$$x_1 = \gamma_1 \cos\theta_1 \sinh\eta_1 = \gamma_1\mu_1\lambda_1,$$

$$\rho_1 = \gamma_1 \sin\theta_1 \cosh\eta_1 = \gamma_1(1-\mu_1^2)^{1/2}(\lambda_1^2+1)^{1/2}.$$

Therefore (I) becomes

$$P_n(\mu_2)Q_n(i\lambda_2) = (-1)^{n+1}\sum_{m=0}^{\infty}(-1)^m(2m+1)\hat{\omega}_1'(m,n)P_m(\mu_2)P_m(i\lambda_2), \quad (B)$$

where

$$\hat{\omega}_1'(m,n) = (-1)^n\frac{2^n n!}{(2n+1)!}\sigma_1'^{\,n+1}\left[\frac{d^n}{dt_1^n}+\frac{\sigma_1'^2}{2(2n+3)}\frac{d^{n+2}}{dt_1^{n+2}}+\cdots\right]Q_m(t_1)$$

$\sigma_1' = \gamma_2/\gamma_1$ and $t_1 = s/i\gamma_1$.

(A) and (B) are the lemmas that will be needed in the discussion of the problems that we shall presently take up.

2. We can put the expressions for $\hat{\omega}_1(m, n)$ and $\hat{\omega}_1(m, n)$ into more compact forms as definite integrals. For

$$\frac{d^rQ_m(t)}{dt^n} = \frac{(-1)^n n!}{2}\int_{-1}^{+1}\frac{P_m(p)dp}{(t-p)^{n+i}}.*$$

Therefore

$$\hat{\omega}_1(m,n) = \frac{2^{n-1}n!\,n!}{(2n+1)!}$$

$$\times\int_{-1}^{+1}\left[\frac{\sigma_1^{n+1}}{(t-p)^{n+1}}+\frac{(n+1)(n+2)}{2(2n+3)}\frac{\sigma_1^{n+3}}{(t-p)^{n+3}}+\cdots\right]P_m(p)dp$$

$$= \frac{1}{2}\int_{-1}^{+1}Q_n\left(\frac{t-p}{\sigma_1}\right)P_m(p)dp.$$

Hence

$$\hat{\omega}_1(m,n) = \frac{1}{2}\int_{-1}^{+1}Q_n\left(\frac{s-x_1p}{k_2}\right)P_m(p)dp.$$

Similarly we obtain

$$\hat{\omega}_1'(m,n) = \frac{1}{2}\int_{-1}^{+1}Q_n\left(\frac{s-i\gamma_1p}{i\gamma_2}\right)P_m(p)dp.$$

§ 2. HYDRODYNAMICAL APPLICATION.

1. Let a_1, c_1 ($a_1 > c_1$) and a_2, c_2 ($a_2 > c_2$) be the semi-axes of two *ovary spheroids* having their centers at O_1 and O_2 respectively, O_1O_2 being the direction of motion.

Let u_1 be the velocity of O_1 towards O_2, u_2 that of O_2 towards O_1; also let s denote the distance between the centers O_1O_2 at any instant. Take λ_1, μ_1, ω and λ_2, μ_2, ω two systems of ovary spheroidal coördinates having their origins at O_1, O_2 respectively, so that the surfaces of the given spheroids are $\lambda_1 = \lambda_{10}$ and $\lambda_2 = \lambda_{20}$.

* Todhunter, *loc. cit.*

The problem before us is to find a function ϕ satisfying the following conditions:

(1) $\nabla^2\phi = 0$, throughout the liquid,

(2) $\phi = 0$, at infinity

(3) $$\frac{\partial\phi}{\partial\lambda_1} = -u_1k_1P_1(\mu_1), \text{ when } \lambda_1 = \lambda_{10},$$

(4) $$\frac{\partial\phi}{\partial\lambda_2} = -u_2k_2P_1(\mu_2), \text{ when } \lambda_2 = \lambda_{20},$$

where $k_1^2 = a_1^2 - c_1^2$ and $k_2^2 = a_2^2 - c_2^2$.

A value of ϕ which satisfies the first two conditions is given by

(5) $$\phi = \sum_{n=1}^{\infty}[A_nP_n(\mu_1)Q_n(\lambda_1) + B_nP_n(\mu_2)Q_n(\lambda_2)].$$

If by a proper choice of the arbitrary constants A_n, B_n, we can make it satisfy the other two conditions, then ϕ will be the required solution.

By the Lemma (A), near the surface of O_1, we have

$$\phi = \sum_{n=1}^{\infty}\left[A_nP_n(\mu_1)Q_n(\lambda_1) + B_n\sum_{m=0}^{\infty}(2m+1)\hat{\omega}_1(m, n)P_m(\mu_1)P_m(\lambda_1)\right].$$

Substituting in the condition (3), we get

$$-u_1k_1P_1(\mu_1) = \sum_{n=1}^{\infty}\Bigg[A_nP_n(\mu_1)Q_n'(\lambda_{10}) + B\sum_{m=}^{\infty}(2m+1)\hat{\omega}_1(m,n)P_m(\mu_1)P_m'(\lambda_{10})\Bigg],$$

where $Q_n'(\lambda_{10})$ and $P_n'(\lambda_{10})$ stand for the values of $(\partial/\partial\lambda_1)Q_n(\lambda_1)$ and $(\partial/\partial\lambda_1)P_n(\lambda_1)$ when λ_{10} has been substituted for λ_1. This must be true at every point on the surface of the spheroid. Hence we can equate to zero the coefficients of the various zonal harmonics of μ_1.

Equating the coefficients of $P_1(\mu_1)$

$$-u_1k_1 = A_1Q_1'(\lambda_{10}) + 3\sum_{n=1}^{\infty}\hat{\omega}_1(1, n)B_n. \tag{6}$$

Equating the coefficients of $P_p(\mu_1)$, $p > 1$,

$$0 = A_pQ_p'(\lambda_{10}) + (2p+1)P_p'(\lambda_{10})\sum_{n-1}^{\infty}\hat{\omega}_1(p, n)B_n. \tag{7}$$

We shall have two similar equations from the remaining surface condition which can, however, be written down from symmetry. Thus

$$- u_2k_2 = B_1Q_1'(\lambda_{20}) + 3\sum_{n=1}^{\infty}\hat{\omega}_2(1, n)A_n, \tag{8}$$

for $p > 1$

$$0 = B_pQ_p'(\lambda_{20}) + (2p+1)P_p'(\lambda_{20})\sum_{n=1}^{\infty}\hat{\omega}_2(p, n)A_n, \tag{9}$$

where $\hat{\omega}_2(m, n)$ can be written, by symmetry, from $\hat{\omega}_1(m, n)$, viz.

$$\hat{\omega}_2(m, n) = \frac{1}{2}\int_{-1}^{+1} Q_n\left(\frac{s - k_2p}{k_1}\right)P_m(p)dp.$$

Thus we have four equations (6) $\cdots$ (9) for the determination of the unknown constants A and B.

2. *The General Values of the Constants A and B.*—Substituting the values of B_1 and B_n $(n > 1)$ from the equations (8) and (9) in the equation (6), we get

$$- u_1k_1 = A_1Q_1'(\lambda_{10}) - 3\frac{\hat{\omega}_1(1, 1)}{Q_1'(\lambda_{20})}\left[u_2k_2 + 3\sum_{n=1}^{\infty}\hat{\omega}_2(1, n)A_n\right]$$
$$- 3\sum_{m=2}^{\infty}(2m+1)\omega_1(1, m)\frac{P_m'(\lambda_{20})}{Q_m'(\lambda_{20})}\sum_{n=1}^{\infty}A_n\hat{\omega}_2(m, n),$$

or

$$\sum_{n=1}^{\infty}A_n\sum_{m=1}^{\infty}(2m+1)\hat{\omega}_1(1, m)\hat{\omega}_2(m, n)\frac{P_m'(\lambda_{20})}{Q_m'(\lambda_{20})}$$
$$- \tfrac{1}{3}A_1Q_1(\lambda_{10}) = \tfrac{1}{3}u_1k_1 - u_2k_2\frac{\hat{\omega}_1(1, 1)}{Q_1(\lambda_{20})}.$$

This equation can be written in the form

$$\sum_{n=1}^{\infty}\theta_{nm}A_n - \tfrac{1}{3}A_1Q_1'(\lambda_{10}) = c, \qquad (m = 1, 2, 3, \cdots \text{ad inf}) \tag{1}$$

where

$$\theta_{nm} = (2m+1)\hat{\omega}_1(1, m)\hat{\omega}_2(m, n)P_m'(\lambda_{20})/Q_m'(\lambda_{20}), \tag{2}$$

and

$$c = \tfrac{1}{3}u_1k_1 - u_2k_2\hat{\omega}_1(1, 1)/Q_1'(\lambda_{20}). \tag{3}$$

In a similar manner, for the determination of B's, we get the equations

$$\sum_{n=1}^{\infty}\theta_{nm}'B_n - \tfrac{1}{3}B_1Q_1'(\lambda_{20}) = c', \qquad (m = 1, 2, 3, \cdots \text{ad inf.}) \tag{4}$$

where

$$\theta_{nm}' = (2m+1)\hat{\omega}_2(1, m)\hat{\omega}_1(m, n)P_m'(\lambda_{10})/Q_m'(\lambda_{10}), \tag{5}$$

and

$$c' = \tfrac{1}{3}u_2k_2 - u_1k_1\hat{\omega}_2(1, 1)/Q_1'(\lambda_{10}). \tag{6}$$

Thus we have finally an infinite number of equations (1) and (4) for the determination of an infinite number of unknowns, A as well as B.

The theory of the solution of the equations of this class has been worked out by Hill, Poincare, Koch, Toeplitz, Schmidt and others. Hence the values of the constants can be found so that the problems becomes determinate, the value of the potential function at any point being given by (5) Art. 1, § 2.

The general values of A as well as B, so determined will, however, be of little help for numerical calculations which will be necessary so as to get a physical idea of the state of liquid at any point. So we give a method of approximating to their values.

3. *Approximate Values of the Constants A and B.*—We have

$$\hat{\omega}_1(m, n) = (-1)^n \frac{2^n n!}{(2n+1)!}\sigma_1^{n+1}\left[\frac{d^n}{dt^n} + \frac{\sigma_1^2}{2(2n+3)}\frac{d^{n+2}}{dt^{n+2}} + \cdots\right] Q_m(t).$$

Substituting the value of $Q_m(t)$ in series, viz.

$$Q_m(t) = \frac{2^m m!\, m!}{(2m+1)!}\left[\frac{1}{t^{m+1}} + \frac{(m+1)(m+2)}{2(2m+3)}\frac{1}{t^{m+3}} + \cdots\right]$$

and rearranging the terms we get

$$\hat{\omega}_1(m, n) = \frac{2^{m+n} m!\, n!\, (m+n)!}{(2m+1)!\,(2n+1)!}\frac{k_1^m k_2^{n+1}}{s^{m+n+1}}$$
$$+ \frac{2^{m+n} m!\, n!\, (m+n+2)!}{(2m+1)!\,(2n+1)!\,2(2n+3)}\frac{k_1^m k_2^{n+1}}{s^{m+n+3}}(k_1^2 + 2k_2^2)$$
$$+ \cdots .$$

Thus the lowest order of $\hat{\omega}_1(m, n)$ is

$$\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^{m+n+1}.$$

If the spheroids are so separated that we can neglect the terms of the order $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$, we get from the equation (1), Art. 1, § 2

$$A_1 = -\frac{u_1 k_1}{Q_1'(\lambda_{10})}; \quad \text{consequently} \quad B_1 = -\frac{u_2 k_2}{Q_2'(\lambda_{20})}, \quad \text{by symmetry.}$$

We see generally that, for $p > 1$

$$A_p = -\frac{2^{p+1} p!\,(p+1)!}{3!\,(2p)!}\frac{P_p'(\lambda_{10})}{Q_p'(\lambda_{10})}\frac{k_1^p k_2^2}{s^{p+2}} B_1 + \text{terms of higher orders,}$$

$$B_p = -\frac{2^{p+1} p!\,(p+1)!}{3!\,(2p)!}\frac{P_p'(\lambda_{20})}{Q_p'(\lambda_{20})}\frac{k_2^p k_1^2}{s^{p+2}} A_1 + \text{terms of higher orders.}$$

We proceed to approximate to the values of the A's and B's

First Approximation.—

$$A_2 = -\frac{2}{3}\frac{P_2'(\lambda_{10})}{Q_2'(\lambda_{10})}\frac{k_1^2k_2^2}{s^4}B_1, \qquad B_2 = -\frac{2}{3}\frac{P_2'(\lambda_{20})}{Q_2'(\lambda_{20})}\frac{k_2^2k_1^2}{s^4}A_1,$$

$$A_3 = -\frac{8}{15}\frac{P_3'(\lambda_{10})}{Q_3'(\lambda_{10})}\frac{k_1^3k_2^2}{s^5}B_1, \qquad B_3 = -\frac{8}{15}\frac{P_3'(\lambda_{20})}{Q_3'(\lambda_{20})}\frac{k_2^3k_1^2}{s^5}B_1,$$

. . . .

Second Approximation.—

$$A_1 = -\frac{u_1k_1}{Q_1'(\lambda_{10})} + \frac{2}{3}\frac{1}{Q_1'(\lambda_{10})Q_1'(\lambda_{20})}\frac{k_2k_2^3}{s^3}u_2,$$

$$B_1 = -\frac{u_2k_2}{Q_1'(\lambda_{20})} + \frac{2}{3}\frac{1}{Q_1'(\lambda_{20})Q_1'(\lambda_{10})}\frac{k_2k_1^3}{s^3}u_1,$$

. . . .

and so on.

4. *Verification of the Results Previously Obtained.*—Neglecting terms of the order $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$ and higher, we have found that

$$\phi = -\frac{u_1k_1}{Q_1'(\lambda_{10})}P_1(\mu_1)Q_1(\lambda_1) - \frac{u_2k_2}{Q_1'(\lambda_{20})}P_1(\mu_2)Q_1(\lambda_2). \tag{1}$$

Expressed in terms of spherical harmonics

$$P_1(\mu_1)Q_1(\lambda_1) = \frac{2}{3}k_1^2\left\{\frac{P_1(\cos\theta_1)}{r_1^2} + \frac{3}{5}k_1^2\frac{P_3(\cos\theta_1)}{r_1^4} + \cdots\right\},$$

$$P_1(\mu_2)Q_1(\lambda_2) = \frac{2}{3}k_2^2\left\{\frac{P_1(\cos\theta_2)}{r_2^2} + \frac{3}{4}k_2^2\frac{P_3(\cos\theta_2)}{r_2^4} + \cdots\right\};$$

also

$$Q_1'(x) = -\frac{2}{3}\left\{\frac{2}{x^3} + \frac{3}{5}\frac{4}{z^5} + \cdots\right\}.$$

Now if e_1 and e_2 be the eccentricities of the generating ellipses of the two spheroids respectively, we know

$$e_1 = 1/\lambda_{10}, \qquad e_2 = 1/\lambda_{20}, \qquad k_1 = a_1e_1, \qquad k_2 = a_2e_2.$$

Substituting in the equation (1) and neglecting powers of e_1 and e_2 higher than the second, we get as far as the required order

$$\phi = \tfrac{1}{2}u_1a_1^3(1 - \tfrac{6}{5}e_1^2)\frac{P_1(\cos\theta_1)}{r_1^2} + \tfrac{3}{10}u_1a_1^5e_1^2\frac{P_3(\cos\theta_1)}{r_1^4}$$
$$+ \tfrac{1}{2}u_2a_2^3(1 - \tfrac{6}{5}e_2^2)\frac{P_1(\cos\theta_2)}{r_2^2} + \tfrac{3}{10}u_2a_2^5e_2^2\frac{P_3(\cos\theta_2)}{r_2^4}.$$

Now from the results of the previous paper, we have neglecting powers of e_1 and e_2 higher than the second

$$A_1 = -\tfrac{1}{2}u_1a^3\left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right), \quad B_1 = -\tfrac{1}{2}u_2b^3\left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right).$$

Making allowance for the difference of notations between the two papers we have

$$a = a_1(1-\tfrac{1}{3}e_1^2), \quad b = a_2(1-\tfrac{1}{3}e_2^2), \quad \epsilon_1 = \tfrac{1}{3}e_1^2, \quad \epsilon_2 = \tfrac{1}{3}e_2^2.$$

Therefore

$$A_1 = -\tfrac{1}{2}u_1a_1^3(1-\tfrac{6}{5}e_1^2), \quad B_1 = -\tfrac{1}{2}u_2a_2^3(1-\tfrac{6}{5}e_2^2).$$

Since the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ are neglected,

$$A_2 = 0, \quad A_3 = -\tfrac{1}{20}u_1a_1^5e_1^2, \quad A_4 = 0, \quad \cdots,$$

$$B_2 = 0, \quad B_3 = -\tfrac{1}{20}u_2a_2^5e_2^2, \quad B_4 = 0, \quad \cdots.$$

Further

$$\frac{d^3}{ds_{12}^3}\left(\frac{1}{r_1}\right) = -3!\,\frac{P_3(\cos\theta_1)}{r_1^4}.$$

Hence the equation (v) becomes up to the order of terms retained

$$\phi = \tfrac{1}{2}u_1a_1^3(1-\tfrac{6}{5}e_1^2)\frac{P_1(\cos\theta_1)}{r_1^2} + \tfrac{3}{10}u_1a_1^5e_1^2\frac{P_3(\cos\theta_1)}{r_1^4}$$
$$+ \tfrac{1}{2}u_2a_2^3(1-\tfrac{6}{5}e_2^2)\frac{P_1(\cos\theta_2)}{r_2^2} + \tfrac{3}{10}u_2a_2^5e_2^2\frac{P_3(\cos\theta_2)}{r_2^4},$$

so that the results of the two papers agree at least up to this order. Similarly it can be shown that the two results agree in their most general form.

It may also be pointed out that, considered term for term, the series for the velocity potential function now obtained is more convergent than the series obtained in the previous paper.

5. *Conclusion.*—In the preceding articles, I have studied the case when the two spheroids have the same common axis of revolution which coincides also with the direction of motion. In a similar way, can be investigated the case when the axes of revolution of the two spheroids are at right angles to the direction of motion and the transformation theorem for spheroidal harmonics that will be required can be obtained.

UNIVERSITY OF CALCUTTA, 1919.

INTEGRAL PRODUCTS AND PROBABILITY.

By P. J. Daniell.

1. *Introduction.*—In many problems arising in statistical biology and statistical economics time enters as an indispensable factor. It is the chief aim of this paper to provide a form of analysis suitable for such problems, and this will be a theory of probability using time as an auxiliary variable. The term "Dynamic Probability" may be used to avoid confusion with Gibbs' Statistical Mechanics, which is a theory of interactions between fast-moving molecules or between vibratory systems, whereas our theory is that of time-variations in position of groups moving more or less erratically. That is to say, the basis is change of position rather than of velocity, and the tendency to disperse is regarded as inherent, not as the effect of "collisions."

For the sake of simplicity we consider only motion in a single dimension. The variable x may be a measure of actual geometrical position, or it may measure some factor of the relative environment, for example, temperature, intensity of light, or it may refer to a quantity of goods.

The first step in the analysis is a search for some standard formula on which may be built a more complex and general theory. It is found that, if certain natural assumptions are made, a functional equation is satisfied, which is expressed in terms of a Stieltjes integral product. This form of product may be compared with Volterra's integral composition, and on it a similar algebra may be built. It is shown that there exists an idem-factor function for this algebra whereas such a *function* does not exist for the Volterra composition.* The Stieltjes integral product itself forms a secondary nucleus for our paper; its investigation will be found in paragraph 3.

2. *Fundamental Equation.*—Denote an increase (possibly negative) in x by y and an increase in time t by u. The motion in the interval u will depend partly on the previous history H of the member considered. Denote by

$$P(y_1, u_1, H_0)$$

* V. Volterra, "Leçons sur les fonctions de lignes." Paris (1913). G. C. Evans, Cambridge Colloquium, New York (1916), p. 117. By introducing an auxiliary algebraic symbol j, Evans modifies the Volterra algebra so as to contain an idem-factor but this factor, $1 + j0$, is independent of the variables. For a resumé of the Stieltjes integral and references the reader is referred to T. H. Hildebrandt, *Bulletin of the American Mathematical Society* (1917), 24, p. 117 and p. 177.

the probability that, in an interval u_1, the increase in x does not exceed y_1 (*i.e.*, $y \leq y_1$) for a certain history H_0 up to time t_0. Similarly we may denote by

$$P(y_2, u_2, H_1)$$

the probability that, in a succeeding interval u_2, the increase in x does not exceed y_2. H_1 will include both H_0 and the motion during the interval u_1.

This is the critical point at which the dynamics of material and living objects diverge. Theoretical mechanics is expressed in terms of velocities and accelerations which are instantaneous time-derivatives. In such a theory, however small we may choose the interval u_1, the second probability $P(y, u_2, H_1)$ depends essentially on the motion during u_1 rather than on H_0. If only u_1 is chosen sufficiently small the contrary is true of living objects. It is to be understood that we are not concerned with the ultimate hypotheses of materialism and vitalism; we regard each object as a whole and do not pretend to delve into the dynamics of its finest constituents which may or may not be of the material type. Our point will be made clear by an illustration. Consider a salmon ascending a stream and suddenly brought to rest by a glass plate, inserted in the stream for a brief instant. After the plate is removed the salmon will resume its motion almost as if there had been no interruption. It will be influenced more by previous history or habit than by the interruption, provided the latter is short. Compare this case with that of a cylinder rolling along a horizontal plane. Even after the very shortest interruption the cylinder will remain at rest. It is more influenced by the immediately preceding history. An example which is an apparent exception is even more illuminating. In a speculative stock market the movement of a particular stock depends partly on the preceding movement. But even here the effect is influenced by the events of some hours or minutes rather than by those of the last hundredth of a second, let us say.

We shall assume that if the interval u_1 is sufficiently small, the probability function P depends on H_0 only, so that

$$P(y_2, u_2, H_1) = P(y_2, u_2, H_0).$$

If we confine our attention to a homogeneous group of members having practically the same history H_0, up to the time t_0, then we may omit the symbol H_0 and define $P(y, u)$ as the probability that the increase in x does not exceed y in a short interval of time u not far removed from a fixed point of time t_0.

This is the fundamental assumption on which our analysis is based; and it is this which sharply distinguishes dynamic probability from the classical theories of mechanics, statistical or otherwise. After an investiga-

tion of $P(y, u)$ in which u is assumed to be small it is possible to build up a more general theory suitable for longer intervals, somewhat as organic chemistry is built on the atomic laws of simpler reactions.

By its definition $P(y, u)$ is a limited non-decreasing function of y approaching 0 as y approaches $-\infty$, and 1 as y approaches $+\infty$. It is also "continuous on the right," that is

$$P(y, u) = \lim_{e \doteq 0,\, e>0} P(y + e, u)$$

$$= P(y + 0, u).$$

On the other hand,

$$P(y - 0, u) = \lim_{e \doteq 0,\, e>0} P(y - e, u)$$

is the probability that the increase in x is *strictly less* than y, and it is not necessarily equal to $P(y, u)$.

The increase y in the interval $u_1 + u_2$ is the sum of an increase z during u_1 and an increase $y - z$ during u_2. Keeping y fixed the function $P(y - z, u_2)$ is a limited non-increasing function of z continuous on the left with the values 1 at $-\infty$, 0 at $+\infty$, and is continuous in each of a countable set of intervals which are open below and closed above.

Let cd be one of these intervals; then the probability that, in the time u_1, the increase z in x satisfies $c < z \leq d$ will be

$$\Delta P(z, u_1) \equiv P(d, u_1) - P(c, u_1).$$

Given that $c < z \leq d$, the probability that the increase in the time $u_1 + u_2$ does not exceed y will lie between

$$P(y - d, u_2),\ P(y - c, u_2).$$

Then the combined probability will lie between

$$P(y - d, u_2)\Delta P(z, u_1),\ P(y - c, u_2)\Delta P(z, u_1).$$

The law of composite probability is applicable here because we assumed the motion during u_2 independent of that in u_1.

Divide cd into smaller sub-intervals and proceed to the limit as their maximum length approaches 0. The combined probability that $c < z \leq d$ and that during $u_1 + u_2$ the increase does not exceed y will be

$$\int_c^d P(y - z, u_2)\, d_z P(z, u_1).$$

Add up the corresponding expressions for all the intervals cd, and then the fundamental equation is obtained, namely

2(1) $$P(y, u_1 + u_2) = \int_{-\infty}^{+\infty} P(y - z, u_2)\, d_z P(z, u_1).$$

Without changes in the processes of reasoning we may regard u_2 as the earlier, u_1 the later interval of time and so obtain

2(2) $$P(y, u_1 + u_2) = \int_{-\infty}^{+\infty} P(y - z, u_1) d_z P(z, u_2).$$

3. *Stieltjes Integral Products.*—The function $\alpha(x)$ will be said to be a regular function of limited variation in x if it is of limited variation $(-\infty$ to $+\infty)$, if $\alpha(-\infty) = 0$ and if, for all values of x,

$$\alpha(x) = \tfrac{1}{2}[\alpha(x + 0) + \alpha(x - 0)].$$

We shall require an important lemma.

3(1). If $\alpha(s, t)$ is a regular function of limited variation in s, and measurable in the sense of Borel in t, and if the variation be limited uniformly with respect to t, and if $g(t)$ is of limited variation and $f(s)$ is bounded and measurable,* then

$$\int_{-\infty}^{+\infty} f(s) d_s \int_{-\infty}^{+\infty} \alpha(s, t) g d(t) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(s) d_s \alpha(s, t) \right] dg(t).$$

Let $f(s) = 1$, $(c \leq s \leq d) = 0$ otherwise.

$$\int_{-\infty}^{+\infty} f(s) d_s \int_{-\infty}^{+\infty} \alpha(s, t) dg(t) = \int_{-\infty}^{+\infty} \alpha(d + 0, t) dg(t) - \int_{-\infty}^{+\infty} \alpha(c - 0, t) dg(t)$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(s) d_s \alpha(s, t) \right] dg(t).$$

If $f(s)$ is a step-function (constant over each of a finite set of sub-intervals) it is a linear combination of functions of the above type and therefore the equality will be satisfied. But any measurable function $f(s)$ can be obtained from such step-functions by a finite succession of linear combinations and limiting processes using in all cases bounded sets of functions since $f(s)$ is bounded. Because $\alpha(s, t)$ is of uniformly limited variation a number S exists such that the total variation of $\alpha(s, t)$ does not exceed S. If G is the variation of $g(t)$ then the variation of $\int_{-\infty}^{+\infty} \alpha(s, t) dg(t)$ does not exceed SG. Also if $|f(s)| \leq M$,

$$\left| \int_{-\infty}^{+\infty} f(s) d_s \alpha(s, t) \right| \leq MS.$$

Therefore in each of the required limiting processes the several integrands do not exceed summable functions in absolute value, and at each step the equality is preserved.†

* In this paper the attribute 'measurable' will be taken in the sense of Borel.

† P. J. Daniell, *Annals of Mathematics*, Vol. 19 (1918), p. 290. The theorem used here is used frequently in the present paper.

hypothesis $\eta' \cdot \eta = \eta$ so that $\eta' = \eta$. Similarly it can be shown that there is no other idem-fector η'' satisfying the conditions in 3(2) and such that $\alpha \cdot \eta'' = \alpha$ for every α.

Thus η is the only idem-factor in the class of functions α considered.

If it happens that $\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_1$ we may say that the functions are S-permutable to distinguish the property from that of being permutable according to Volterra.

If α_1, α_2 are differentiable with respect to their first variables, for example

$$\alpha_1(s, t) = \int_{-\infty}^{s} f_1(s, t)ds, \qquad \alpha_2(s, t) = \int_{-\infty}^{s} f_2(s, t)ds,$$

then

$$\alpha_1 \cdot \alpha_2(s, t) = \int_{-\infty}^{s} \overset{*}{f_1}\overset{*}{f_2}(s, t)ds,$$

in the Volterra notation, and this shows the relation between the two types of combination.

A development of this algebra particularly in connection with integral equations would be interesting but we shall now confine our attention to functions of the difference $x - y$,

$$\alpha(x, y) = \alpha(x - y).$$

Then

$$\alpha_1 \cdot \alpha_2 = \int_{-\infty}^{+\infty} \alpha_1(x - s)d_s\alpha_2(s - y) = \int_{-\infty}^{+\infty} \alpha_1(x - y - z)d_z\alpha_2(z),$$

so that $\alpha_1 \cdot \alpha_2$ is also a function of the difference $x - y$.

3(3). To use symbols more in keeping with our special problem, if $\alpha_1(y)$, $\alpha_2(y)$ are regular functions of limited variation ($-\infty$ to $+\infty$), then we define their $S - V$ product as

$$\alpha_1 \cdot \alpha_2(y) \equiv \int_{-\infty}^{+\infty} \alpha_1(y - z)d\alpha_2(z) \quad \textit{Def.}$$

The corresponding idem-factor will be

$$\begin{aligned} \eta(y) &= 0, \quad y < 0, \\ &= \tfrac{1}{2}, \quad y = 0, \\ &= 1, \quad y > 0. \end{aligned}$$

3(4). THEOREM. *The product as defined in* 3(3) *is S-permutable. For since the functions are regular, by a theorem on integration by parts,**

* P. J. Daniell, *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 362.

4(2). If

$$\int_{-\infty}^{+\infty} e^{hz} d\beta(z), \qquad -H < h < +K,$$

exists, where H, K are *positive*, then so also does

$$\int_{-\infty}^{+\infty} z^n e^{hz} d\beta(z), \qquad -H < h < +K, \qquad n = 1, 2, \cdots.$$

For if $-H < h < K$ we can find a number $\eta > 0$ such that $-H < h - \eta < h + \eta < K$. Given η and n we can find $z_0 > 1$ such that $z^n < e^{\eta z}$, $z \geq z_0$.

Then $z^n e^{hz}(z \geq 0)$ is measurable and less than the function $z_0^n e^{(h+\eta)z}$ which is summable β from $-\infty$ to $+\infty$.

Similarly $z^n e^{hz}(z \leq 0)$ is measurable and its modulus is less than $z_0^n e^{(h-\eta)z}$ which is summable β from $-\infty$ to $+\infty$. Therefore $z^n e^{hz}$ is summable β from $-\infty$ to $+\infty$.

4(3). If

$$\int_{-\infty}^{+\infty} e^{hz} d\beta(z), \qquad -H < h < K,$$

exists then

$$\int_{-\infty}^{+\infty} e^{pz} d\beta(z) = \varphi(p, \beta)$$

is holomorphic in the strip $-H <$ real part of $p < K$. This strip will be called the HK strip.

If $p = h + ik$ lies in this strip, $-H < h < K$ and since $|\cos kz|$, $|\sin kz|$ are not greater than 1, $|e^{hz}| = e^{hz}$, it follows that

$$\varphi(p, \beta) = \int_{-\infty}^{+\infty} e^{hz} \cos kz \, d\beta(z) + i \int_{-\infty}^{+\infty} e^{hz} \sin kz \, d\beta(z)$$

exists.

$$[\varphi(p + \Delta p) - \varphi(p)]/\Delta p = \int_{-\infty}^{+\infty} z e^{pz} \psi(z\Delta p) d\beta(z)$$

in which $\psi(t) = (e^t - 1)/t$.

Integrating along a radius $\int_0^t e^t dt = e^t - 1$, so that $|e^t - 1| \leq |t| e^{|t|}$, and $|\psi(t)| < e^{|t|}$.

Since $-H < h < K$ we can find a number $\eta > 0$ such that $-H < h - \eta < h + \eta < K$ and if $|\Delta p| \leq \eta$, the real and imaginary parts of $ze^{pz}\psi(z\Delta p)$ will not be greater in modular values than the function which has for each z the greater of the values of $|z| e^{(p \pm \eta)z}$, which by 4(2) is known to be summable.

Furthermore $\psi(t)$ is continuous at $t = 0$ and its limit there is 1. Therefore by the theorem used in proving 3(1) there exists a unique limit of

$\Delta\varphi/\Delta p$ as Δp approaches 0, that is to say, $\varphi(p)$ is holomorphic in the HK strip and

$$\frac{d\varphi(p)}{dp} = \int_{-\infty}^{+\infty} ze^{pz}d\beta(z).$$

4(4). If $\varphi(h, \beta_1)$, $\varphi(h, \beta_2)$ exist, $-H < h < K$ and if $\beta_1 \cdot \beta_2(y) = \gamma(y)$ then

$$\varphi(p, \gamma) = \varphi(p, \beta_1)\varphi(p, \beta_2)$$

provided p is in the HK strip and conversely.

$$\begin{aligned}\varphi(p, \gamma) &= \int_{-\infty}^{+\infty} e^{py}d_y \int_{-\infty}^{+\infty} \beta_1(y - z)d\beta_2(z) \\ &= \lim_{Y=\infty} \int_{-Y}^{+Y} e^{py}d_y \int_{-\infty}^{+\infty} \beta_1(y - z)d\beta_2(z) \\ &= \lim_{Y=\infty} \int_{-\infty}^{+\infty} \left[\int_{-Y}^{+Y} e^{py}d_y\beta_1(y - z)\right] d\beta_2(z)\end{aligned}$$

by theorem 3(1), and this

$$= \lim_{Y=\infty} \int_{-\infty}^{+\infty} e^{pz} \left[\int_{-Y+z}^{Y-z} e^{pv}d\beta_1(v)\right] d\beta_2(z).$$

The expression in square brackets is not greater in modular value than $\varphi(h, \omega_1)$ where h is the real part of p, $\omega_1(v)$ is the modular variation function corresponding to $\beta_1(v)$. The existence of $\varphi(h, \beta_1)$ implies that e^{hv} is summable with respect to β_1, and therefore with respect to the variation function ω_1. Therefore using again the theorem with respect to limits under the integral sign

$$\begin{aligned}\varphi(p, \gamma) &= \int_{-\infty}^{+\infty} e^{pz}\varphi(p, \beta_1)d\beta_2(z) \\ &= \varphi(p, \beta_1)\varphi(p, \beta_2).\end{aligned}$$

To prove the converse let $\delta(y) = \beta_1 \cdot \beta_2(y)$, then by the theorem itself

$$\begin{aligned}\varphi(p, \gamma - \delta) &= \varphi(p, \gamma) - \varphi(p, \delta) \\ &= \varphi(p, \gamma) - \varphi(p, \beta_1)\varphi(p, \beta_2) \\ &= 0 \text{ by hypothesis.}\end{aligned}$$

Therefore by 4(1). Cor. $\gamma - \delta = 0$ identically, and $\gamma(y) = \beta_1 \cdot \beta_2(y)$.

4(5). If $\beta_u(y)$ is a regular function of limited variation in y for $u \geqq 0$, if $\beta(y)$ is not identically 0 and if $\varphi(p, \beta_u)$ exists in the HK strip, then if $\varphi(p, \beta_u) = e^{uP}$, where P is holomorphic in the HK strip, and where u is rational and non-negative,

$$\beta_{u_1}\cdot\beta_{u_2}(y) = \beta_{u_1+u_2}(y)$$

and conversely.

The direct theorem is a simple consequence of 4(4) since

$$\varphi(p,\ \beta_{u_1+u_2}) = e^{(u_1+u_2)P} = \varphi(p,\ \beta_{u_1})\varphi(p,\ \beta_{u_2}).$$

To prove the converse let $p = \log_e \varphi(p,\ \beta_1)$, i.e., where $u = 1$, then by 4(4) since $\beta_{u_1}\cdot\beta_{u_2}(y) = \beta_{u_1+u_2}(y)$,

$$\log \varphi(p,\ \beta_{u_1}) + \log \varphi(p,\ \beta_{u_2}) = \log \varphi(p,\ \beta_{u_1+u_2}),$$

from which, if u is rational and non-negative, we obtain

$$\log \varphi(p,\ \beta_u) = u \log \varphi(p,\ \beta_1) = uP, \qquad \varphi(p,\ \beta_u) = e^{uP}.$$

To complete the theorem it remains to be proved that P is holomorphic in the HK strip. Since $\varphi(p,\ \beta_1)$ is holomorphic P will be holomorphic except where $\varphi(p,\ \beta_1) = 0$. Let us suppose that at p_0, $\varphi(p_0,\ \beta_1) = 0$ then for all non-negative rational u, $\varphi(p_0,\ \beta_u) = 0$. But if $u = nv$, $\varphi(p,\ \beta_u) = [\varphi(p,\ \beta_v)]^n$ when n is a positive integer, and consequently $\delta^r\varphi(b,\ \beta_u)]_{p=p_0}$, $r < n$, $= 0$, where δ stands for d/dp.

But n can be any positive integer and r any integer less than n so that not only φ but all its derivatives must vanish at p_0. By Taylor series and continuation it follows that $\varphi(p,\ \beta_u) = 0$ identically over all the HK strip (boundaries excluded) and in particular that it vanishes identically along the imaginary axis. But by 4(1). Cor. this implies that $\beta_u(y) = 0$ identically which is a case excluded by hypothesis. This completes the proof of the theorem.

This theorem enables us to pass from the functional equation

3(5) $$\beta_{u_1}\cdot\beta_{u_2} = \beta_{u_1+u_2}$$

to the algebraic relation $\varphi(p,\ \beta_u) = e^{uP}$ and back. The formal solution of 3(5) can now be written in the form

4(6) $$\beta_u(y) = \int_0^\infty \frac{dk}{\pi k}[b^u \sin k - e^{uR(k)} \sin\{kI(k) - ky\}]$$

where $R(k) + iI(k) = P(ik) = \log \varphi(ik,\ \beta_1)$, $b = e^{P(0)} = \varphi(0,\ \beta_1) = \beta_1(\infty)$ $b^u = e^{uP(0)} = \varphi(0,\ \beta_u) = \beta_u(\infty)$.

5. *Particular Solutions.*—In our search for the more elementary solutions we shall carry through some processes in a purely formal manner without rigorous justification, but we can test the results obtained. If it is found that $\varphi(p,\ \beta_u) = e^{uP}$ where P is holomorphic in an HK strip then by 4(5) $\beta_u(y)$ will be a solution of the functional equation 3(5).

In some cases, particularly if $\int_0^\infty e^{uR(k)}dk$ is convergent, we may differentiate both sides of 4(6) and obtain the simplified formula

$$\beta_u(y) = \int_{-\infty}^{y} f_u(y)dy$$

where

$$f_u(y) = \frac{1}{\pi}\int_0^\infty e^{uR(k)} \cos [kI(k) - ky]dk.$$

Let us suppose that $f_u(y)$ satisfies some differential equation which may be written in the form $\Sigma_n\ A_n(D)(y^n f) = 0$, where $D \equiv d/dy$, $A_n(D)$ is a polynomial in the operator D whose coefficients depend on u. Now under certain conditions

$$\int_{-\infty}^{+\infty} e^{pz}z^n f_u(z)dz = \delta^n \int_{-\infty}^{+\infty} e^{pz} f_u(z)dz, \qquad \delta \equiv d/dp,$$

$$= \delta^n e^{uP}.$$

$$\int_{-\infty}^{+\infty} e^{pz}D[Z(z)]dz = [e^{pz}Z]_{-\infty}^{+\infty} - p\int_{-\infty}^{+\infty} e^{pz}Zdz$$

$$= -p\int_{-\infty}^{+\infty} e^{pz}Zdz.$$

$$\int_{-\infty}^{+\infty} e^{pz}A_n(D)[z^n f_u(z)]dz = A_n(-p)\delta^n e^{uP}.$$

Then P must satisfy the differential equation

$$\Sigma_n A_n(-p)\delta^n e^{uP} = 0.$$

This equation is a differential equation for P which is a function of p only and therefore the ratios of the coefficients of any two distinct types of products of P with its derivatives must be independent of u. It follows that the equation can contain only two terms, those for $n = 0$ and $n = 1$. Consider for example the case $n = 2$. Then

$$A_2(-p)\delta^2(e^{uP}) = A_2(-p)e^{uP}[u\delta^2 P + u^2(\delta P)^2]$$

and the ratio of the coefficients of these two terms is u and is *not* independent of u. Making the elimination of such terms there remains

$$A_1(-p)\delta e^{uP} + A_0(-p)e^{uP} = 0,$$

$$uA_1(-p)\delta P + A_0(-p) = 0.$$

Then $A_1(-p) = A(-p)$ will be independent of u while $A_0(-p) = uB(-p)$ where B is also independent of u. Since in the case considered $e^{uP(0)} = \beta_u(\infty) = 1$, $P(0)$ must $= 0$ and

$$P(p) = \int_0^{-p} \frac{B(q)}{A(q)} dq.$$

Correspondingly we have for $f_u(y)$ the equation

$$A(D)(yf) + uB(D)f = 0.$$

In general the simplest form of solution of this equation is given by definite integrals of type 4(6),* so that we do not appear to have made much progress. However if dP/dp is a rational function of p it may be resolved into a sum of terms of types $a_n p^n$, $b_n/(p - p_0)^n$ and if β_u, γ_u correspond to two functions P_1, P_2 there will be a solution $\delta_u = \beta_u \cdot \gamma_u$ corresponding to $P_1 + P_2$ so that more general solutions can be built out of a few simpler types.

Pearson† has suggested types of statistical distributions which are more general than the "normal" or Gaussian distribution. They are such as to satisfy a differential equation of the form

$$\frac{1}{f}\frac{df}{dy} = \frac{a_1 - y}{b_0 + b_1 y + b_2 y^2}.$$

When this equation is cleared of fractions there appears a term $b_2 y^2 Df$ and in such a case f cannot lead to a solution of 3(5). We must therefore choose $b_2 = 0$, and then

$$(b_1 D + 1)(yf) + (b_0 D - a_1 - b_1)f = 0.$$

Comparing with the form already given

$$A(D) = b_1 D + 1, \quad B(D) = (b_0 D - [a_1 + b_1])/u = c_0 D - c_1.$$

$$P(p) = \int_0^{-p} \frac{c_0 q - c_1}{b_1 q + 1} dq.$$

5(1). If $b_1 = 0$, $P = \frac{1}{2}c_0 p^2 + c_1 p$, $Df/f = (c_1 u - y)/c_0 u$ and $\log f = - y^2/2c_0 u + (c_1/c_0)y + C$, where C is some constant, which must be chosen so that $\beta_u(\infty) = \int_{-\infty}^{+\infty} f_u(y)dy = 1$.

In fact this gives the normal distribution

$$f_u(y) = \frac{1}{\sqrt{(2\pi c_0 u)}} e^{-(y - c_1 u)^2/2c_0 u}$$

in which the average is at $c_1 u$ (i.e. moving with velocity c_1), and the standard deviation is $\sqrt{(c_0 u)}$.

* Cf. A. R. Forsyth, "Treatise on Differential Equations" (1903), 3d ed., p. 250.

† K. Pearson, *Philosophical Transactions of the Royal Society*, 186A, p. 343.

$$\varphi(p, \beta_u) = \int_{-\infty}^{+\infty} e^{py} f_u(y) dy$$

$$= \frac{1}{\sqrt{(2\pi c_0 u)}} e^{\frac{1}{2}c_0 u p^2 + c_1 u p} \int_{-\infty}^{+\infty} e^{-(y - c_1 u - c_0 u p)^2/2c_0 u} dy$$

$$= e^{uP}.$$

This proves that here is a possible solution.

5(2). If $b_1 \neq 0$, let $p_0 = 1/b_1$, $n = c_1 p_0 + c_0 p_0^2$, then

$$P(p) = -c_0 p_0 p - n \log(1 - p/p_0),$$

$$\log f = -p_0 y + (nu - 1) \log(y + uc_0 p_0) + C,$$

where C is some constant chosen so that $\beta_u(\infty) = 1$.

From the form of this solution it is evident that $y > -uc_0 p_0$, or in other words that $f_u(y) = 0$, when $y < -uc_0 p_0$.

When $y > -uc_0 p_0$, $f_u(y) = e^{-p_0 y}(y + uc_0 p_0)^{nu-1} p_0^{nu} e^{-uc_0 p_0^2}/\Gamma(nu)$. In this case there is a sharp boundary at $y = -uc_0 p_0$, which is therefore moving with velocity $-c_0 p_0$. The average is at $(nu/p_0) - uc_0 p_0$, the standard deviation is $\sqrt{(nu)}/p_0$.

By choosing C to be partly imaginary, $C = C' + (nu - 1) \log(-1)$, we can obtain another solution in which $f_u(y) = 0$ when $y > -uc_0 p_0$. For the convergence of the integral p_0 must now be negative, $= -q_0$ and then when $y < uc_0 q_0$, $f_u(y) = e^{q_0 y}(uc_0 q_0 - y)^{nu-1} q_0^{nu} e^{-uc_0 q_0^2}/\Gamma(nu)$. This distribution is the mirror image of the other, reflection taking place at $y = -uc_0 p_0$ and with $q_0 = -p_0$ in place of p_0. The two cannot be combined since in one case p_0 is positive, in the other negative. When a comparison is made of the two types of solution we see that type 5(1) is continuous and as soon as u becomes positive there is a finite non-zero density of distribution in both directions; type 5(2) has a discontinuity at a moving boundary and the distribution has non-zero density on one side only of this boundary. Also in 5(2) the rate of decrease in density is less rapid than in 5(1). To make a true comparison of these rates it is necessary to make allowance for the different deviations. Let $\bar{y}$ be the average y, σ the standard deviation and put $y = \bar{y} + r\sigma$ where r is some pure number greater than 1. In case 5(1) $\bar{y} = c_1 u$, $\sigma = \sqrt{(c_0 u)}$, and

$$-\sigma Df/f = \sqrt{(c_0 u)}\,[c_1 u + r\sqrt{(c_0 u)} - c_1 u]/c_0 u$$

$$= r.$$

In case 5(2) $\bar{y} = nu/p_0 - uc_0 p_0$, $\sigma = \sqrt{(nu)}/p_0$, and

$$-\sigma Df/f = \frac{\sqrt{(nu)}}{p_0}\left[p_0 + \frac{1-nu}{nu/p_0 + r\sqrt{(nu)}/p_0}\right]$$

$$\doteq \frac{1 + r\sqrt{(nu)}}{r + \sqrt{(nu)}}.$$

If $r > 1$ this is $< r$ and if r is large while u is small it is approximately $1/r$.

As the next step we might consider differential equations for f which are of the second order, that is in which dP/dp is a fraction in which either the denominator or numerator (or both) is quadratic in p. If the denominator reduces to a constant we have a case similar to 5(1) but in which P contains a term in p^3. The resulting solution for f is not elementary. If the denominator is linear, P is the sum of terms of type 5(1) and 5(2), the resulting solution is an integral composition of 5(1) and 5(2) and is not elementary.

Finally if the denominator is quadratic, P is the sum of types 5(2)and the combination in f is again not elementary.

There are an endless number of other possible solutions but the author has not had the good fortune to find types which should be so simple that they could be used as a basis for further investigation. Even the type 5(2), simple though it is, does not seem to be readily applicable to problems in dynamic probability. It may, however, be useful in exceptional cases.

For the remainder of this paper we shall use the solution 5(1) in the forms: $\beta_u(y) = \int_{-\infty}^{y} f_u(y)dy$, where

5(3) $$f_u(y) = \frac{1}{\sqrt{(2\pi Mu)}} e^{-(y-Uu)^2/2Mu},$$

5(4) $$f_u(y) = \sqrt{(R/2\pi u)}e^{-Ry^2/2u}e^{Fu}e^{-uF^2/2R},$$

$$R = 1/M, \qquad F = RU, \qquad U = MF.$$

U is the velocity of the average change in x, or we may call U the *drift* of the group.

M is the rate of increase of the average square deviation and is called the *mobility*. The standard deviation in an interval u is $\sqrt{Mu}$.

F is the ratio of drift to mobility and is called the *force*.

R is the reciprocal of the mobility and is called the *resistance*.

The formulæ 5(3), 5(4) are really identical but in some cases one or the other form is more convenient and natural.

6. *Variable Characteristics.*—We now consider the motion of a group during finite intervals of time and over finite intervals in the variable. We assume that the motion is the result of infinitesimal movements of the

type 5(3), that is to say, that when u is sufficiently small the motion is as close to that of 5(3) as we choose but that the characteristics U, M (or F, R) vary both with initial position x and initial time t. It is also assumed that, except possibly at a boundary, the first and second partial derivatives with respect to x of these characteristics exist and are limited, and that U, M, F, R are also limited.

Let y refer to a displacement from an initial position x, u a positive increase in the time t and denote $f_u(y)$ by

$$f(y,\,u;\;x,\,t) = \sqrt{(R/2\pi u)}e^{-R(y-Uu)^2/2u},$$

where R, U are possibly functions of x, t. Then

$$\int_{-\infty}^{+\infty} f(y,\,u;\;x,\,t)dy = 1,$$

$$\int_{-\infty}^{+\infty} f(y,\,u;\;x,\,t)ydy = Uu,$$

$$\int_{-\infty}^{+\infty} f(y,\,u;\;x,\,t)y^2dy = Mu,$$

$$\int_{-\infty}^{+\infty} f(y,\,u;\;x,\,t)y^{n+1}dy = 0, \qquad n > 1,$$

if we neglect terms of the second, and higher, order in u.

In an individual case the number of members of the group lying in a given interval will be an integer and the distribution will be completely discontinuous. But our problem is that of probable, or average, not actual distribution and such a probable number may be fractional or even irrational. We assume that the probable number of members of the group in the interval x to $x + dx$ is $N(x, t)dx$, that $N(x, t)$ is limited and summable from $-\infty$ to $+\infty$ and that it possesses bounded first and second partial derivatives with respect to x.

If $K(u)$ is the number crossing the point x in the direction of x increasing,

$$\begin{aligned} K(u) &= \int_{-\infty}^{x} N(s,\,t)ds \int_{x-s}^{\infty} f(y,\,u;\;s,\,t)dy \\ &= \int_{0}^{\infty} dy \int_{x-y}^{x} f(y,\,u;\;s,\,t)N(s,\,t)ds. \end{aligned}$$

This change of order of integration is legitimate since $N(s, t)$ is summable in s and

$$|f(y,\,u;\;s,\,t)| \leq \sqrt{R_1/2u}e^{-R_2y^2/2u}e^{F_1y},$$

where R_1 is the maximum, R_2 the minimum value of R, F_1 the maximum value of $F = RU$, the expression on the right-hand side being independent of s and summable in y.

If $x - p$ is substituted for s we obtain

$$K(u) = \int_0^\infty dy \int_0^y f(y, u; x - p, t) N(x - p, t) dp.$$

Expanding in powers of p,

$$f(y, u; x - p, t) N(x - p, t) = f(y, u; x, t) N(x, t) - p \frac{\partial}{\partial x}[f(y, u; x, t) N(x, t)] + \frac{p^2}{2} \epsilon(y, p),$$

$$|\epsilon(y, p)| \leq \max_{p=0 \text{ to } y} \left| \frac{\partial^2}{\partial x^2}[f(y, u; x - p, t) N(x - p, t)] \right|$$

$$\leq (a_0 + a_1 y^2/u + a_2 y^4/u^2)(1 + b_1 y + b_2 u) \sqrt{(R_1/2u)} e^{-R_2 y^2/2u} e^{F_1 y},$$

where a_0, a_1, a_2, b_1, b_2 are certain constants depending on the bounds of the derivatives of U, M, F, R but are independent of y and u. From this inequality it follows that

$$\left| \int_0^\infty dy \int_0^y \frac{1}{2} p^2 \epsilon(y, p) dp \right| \leq \text{a term of order } u^2.$$

It can be neglected and considering terms of order u, or less,

$$K(u) = \int_0^\infty dy \int_0^y f(y, u; x, t) N(x, t) dp - \int_0^\infty dy \int_0^y \frac{\partial}{\partial x}\left[N(x, t) \int_0^\infty f(y, u; x, t) dp \right]$$

$$= N(x, t) \int_0^\infty y f(y, u; x, t) dy - \frac{1}{2} \frac{\partial}{\partial x}\left[N(x, t) \int_0^\infty y^2 f(y, u; x, t) dy \right].$$

Similarly the number $L(u)$ crossing in the other direction is

$$L(u) = \int_x^\infty N(s, t) ds \int_{-\infty}^{x-s} f(y, u; s, t) dy$$

$$= \int_{-\infty}^0 dy \int_x^{x-y} f(y, u; s, t) N(s, t) ds$$

$$= - N(x, t) \int_{-\infty}^0 y f(y, u; x, t) dy + \frac{1}{2} \frac{\partial}{\partial x}\left[N(x, t) \int_{-\infty}^0 y^2 f(y, u; x, t) dy \right].$$

Combining these two, the net resultant number crossing to the right (x increasing) is $K(u) - L(u)$ or

$$N(x, t) \int_{-\infty}^{+\infty} yf(y, u;\ x, t)dy - \frac{1}{2}\frac{\partial}{\partial x}\left[N(x, t) \int_{-\infty}^{+\infty} y^2 f(y, u;\ x, t)dy \right]$$

$$= NUu - \frac{1}{2}\frac{\partial}{\partial x}(NMu).$$

6(1). The net rate at which members cross at x in the direction of x increasing is given by

$$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM).$$

6(2). Taking into consideration the flow across a neighboring point $x + dx$, if there are no sources or sinks,

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x}(NU) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(NM).$$

This equation is similar to that governing the flow of heat along a bar, but there is an added term depending on drift.

For statistically steady *flow* $\partial N/\partial t = 0$,

$$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) \text{ is independent of } x.$$

For statistical equilibrium (statistically stationary state)

$$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0$$

of which the general solution is $N = (C/M)e^{2\phi}$ where C is an arbitrary constant, φ is the attractive potential defined so that $\partial\varphi/\partial x = F = RU = U/M$, and U, M are now assumed to be independent of t.

It is evident that N tends to be large where the attractive potential is high or where the mobility is small.

In economics φ may be regarded as a possible substitute for "desirability" or "ophelimity."

7. *Boundary Conditions.*—At a rigid boundary there will not only be a resultant 0 flow but the flow will be 0 separately in each direction. Suppose that at x there is a distribution with finite density for smaller values of x and density 0 for larger values of x; then $L(u) = 0$ because $N(s, t) = 0$, $s > x$. Neglecting terms of order higher than u, we have seen that

$$K(u) = N \int_0^{\infty} yfdy - \frac{1}{2}\frac{\partial}{\partial x}\left[N \int_0^{\infty} y^2 fdy \right].$$

Let $y = Uu + z\sqrt{2Mu}$, then

$$\int_0^\infty yf dy = \frac{1}{\sqrt{\pi}}\int_{-U\sqrt{(u/2M)}}^\infty (Uu + z\sqrt{2Mu})e^{-z^2}dz$$

$$= [\sqrt{(Mu/2\pi)} + \tfrac{1}{2}Uu] + \text{terms of higher order in } u.$$

$$\int_0^\infty y^2 f dy = \frac{1}{\sqrt{\pi}}\int_{-U\sqrt{(u/2M)}}^\infty (Uu + z\sqrt{2Mu})^2 e^{-z^2}dz$$

$$= \tfrac{1}{2}Mu + \text{terms of higher order.}$$

Therefore to order u,

$$K(u) \doteq \tfrac{1}{2}u\left[NU - \frac{1}{2}\frac{\partial}{\partial x}(NM)\right] + \sqrt{u}N\sqrt{(M/2\pi)}.$$

This shows that the flow to the right consists of half the net statistical flow which would occur if the distribution were uniform through x, instead of being bounded at x, together with a flow in which the initial quantity flowing in time u is proportional to $\sqrt{u}$. The latter constitutes a sudden gush from the region of finite density into the empty space. At an impassable boundary this gushing flow must also vanish and the necessary conditions will be

$$M = 0, \qquad NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0.$$

The condition $M = 0$ makes $R = \infty$ and this violates the conditions imposed in paragraph 6, so that we cannot conclude immediately that these are also sufficient.

If at the boundary x_0, $M = 0$ every member of the group which happens to arrive at x_0 will have immediately thereafter a velocity $U = U_0$ and the dispersion will be 0. It would appear to be a sufficient condition that U_0 should be non-positive, for then the group would not pass x_0. But since at any point $x < x_0$, there is a flow of rate $NU - \frac{1}{2}(\partial/\partial x)(NM)$ there will be a sudden change in the density between x and x_0 unless this expression also approaches 0 at $x = x_0$. Since $M_0 = 0$, this condition is equivalent to $U_0 = \frac{1}{2}(\partial M/\partial x)_0$. Now M is positive except at $x = x_0$ and if it possesses a derivative at $x = x_0$, $(\partial M/\partial x)_0$ must be non-positive, or U_0 will be non-positive. It therefore appears that the two conditions

$$M = 0, \qquad NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0$$

are sufficient as well as necessary. Let $W = U - \frac{1}{2}(\partial M/\partial x)$, then at $x = x_0$, $M = 0$, $W = 0$. Now

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x}\left[NW - \tfrac{1}{2}M\frac{\partial N}{\partial x}\right] = -N\frac{\partial W}{\partial x} + \left(\frac{1}{2}\frac{\partial M}{\partial x} - W\right)\frac{\partial N}{\partial x} + \tfrac{1}{2}M\frac{\partial^2 N}{\partial x^2}.$$

At $x = x_0$ this becomes

$$\frac{\partial N_0}{\partial t} = -N_0(\partial W/\partial x)_0 + \tfrac{1}{2}(\partial M/\partial x)_0(\partial N/\partial x)_0.$$

It may happen that the boundary produces its effect by a "repulsion" which is considerable at some distance, but consider the case where U, M are practically unaffected until points very near x_0 are reached. Then in general M will suddenly drop to 0 from a finite value and $(\partial M/\partial x)_0$ will be large. If $(\partial N/\partial x)_0$ does not happen to be 0, $\partial N_0/\partial t$ will be large numerically unless there is some relation of the type $\partial N/\partial x = 2kN$, where k is a finite number such that $\partial W/\partial x - k\partial M/\partial x$ remains finite. Now if at the boundary *point* x_0 there were statistical equilibrium without any *real* boundary we should have $\partial N/\partial x = 2(W/M)N$. Choose $k = W'/M'$, where W', M' are the values W, M would have in the absence of a boundary. Since these would not vary very rapidly near x_0, the conditions would be satisfied. Hence we may write the general equations for our form of dynamic probability in the form:

7(1) $$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x}(NU) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(NM)$$

throughout the range, together with the boundary conditions:

7(2) $$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0,$$

in which U, M take the values which they would have in the absence of any boundary. It is important, however, to remember that in a small neighborhood of a boundary U and M actually vary rapidly and that their values will differ from those which they would possess in the absence of a boundary.

Since $W_0 = 0 = U_0 - \tfrac{1}{2}(\partial M/\partial x)_0$, $F_0 = R_0\dot{U}_0 = \dfrac{1}{2}\left(\dfrac{1}{M}\dfrac{\partial M}{\partial x}\right)_0$, there will be a strong repulsive "force" very close to a boundary. The author hopes to obtain some interesting results by an extension of the analysis to two or more dimensions.

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INTRODUCTION TO A GENERAL THEORY OF ELEMENTARY PROPOSITIONS.

BY EMIL L. POST.

INTRODUCTION.

In the general theory of logic built up by Whitehead and Russell* to furnish a basis for all mathematics there is a certain subtheory† which is unique in its simplicity and precision; and though all other portions of the work have their roots in this subtheory, it itself is completely independent of them. Whereas the complete theory requires for the enunciation of its propositions real and apparent variables, which represent both individuals and propositional functions of different kinds, and as a result necessitates the introduction of the cumbersome theory of types, this subtheory uses only real variables, and these real variables represent but one kind of entity which the authors have chosen to call elementary propositions. The most general statements are formed by merely combining these variables by means of the two primitive propositional functions of propositions Negation and Disjunction; and the entire theory is concerned with the process of asserting those combinations which it regards as true propositions, employing for this purpose a few general rules which tell how to assert new combinations from old, and a certain number of primitive assertions from which to begin.

This theory in a somewhat different form has long been the subject matter of symbolic logic.‡ However, although it had reached a high state of development as a theory of classes, it had this incurable defect as a logic of propositions, that it used informally in its proofs the very propositions whose formal statements it tried to prove. This defect appears to be entirely overcome in the development of 'Principia.' But owing to the particular purpose the authors had in view they decided not to burden their work with more than was absolutely necessary for its achievement, and so gave up the generality of outlook which characterized symbolic logic.

It is with the recovery of this generality that the first portion of our paper deals. We here wish to emphasize that the theorems of this paper

* A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. 1, 1910; Vol. 2, 1912; Vol. 3, 1913. Camb. Univ. Press.

† *Ibid.*, Vol. 1, part 1, section A.

‡ See C. I. Lewis, "A Survey of Symbolic Logic," University of California Press, 1918. An extensive bibliography is given there.

are *about* the logic of propositions but are *not included* therein. More particularly, whereas the propositions of 'Principia' are *particular* assertions introduced for their interest and usefulness in later portions of the work, those of the present paper are about the set of *all* such possible assertions. Our most important theorem gives a uniform method for testing the truth of any proposition of the system; and by means of this theorem it becomes possible to exhibit certain general relations which exist between these propositions. These relations definitely show that the postulates of 'Principia' are capable of developing the complete system of the logic of propositions without ever introducing results extraneous to that system—a conclusion that could hardly have been arrived at by the particular processes used in that work.

Further development suggests itself in two directions. On the one hand this general procedure might be extended to other portions of 'Principia,' and we hope at some future time to present the beginning of such an attempt. On the other hand we might take cognizance of the fact that the system of 'Principia' is but one particular development of the theory—particular in the primitive functions it employs and in the postulates it imposes on those functions—and so might construct a general theory of such developments. This we have tried to do in the other portions of the paper. Our first generalization leads to systems which are essentially equivalent to that of 'Principia' and connects up with the work of Sheffer* and Nicod† in reducing the number of primitive functions and of primitive propositions respectively. The second generalization, on the other hand, while including the first also seems to introduce essentially new systems. One class of such systems, and we study these in detail, seems to have the same relation to ordinary logic that geometry in a space of an arbitrary number of dimensions has to the geometry of Euclid. Whether these "non-Aristotelian" logics and the general development which includes them will have a direct application we do not know; but we believe that inasmuch as the theory of elementary propositions is at the base of the complete system of 'Principia,' this broadened outlook upon the theory will serve to prepare us for a similar analysis of that complete system, and so ultimately of mathematics.

Finally a word must be said about the viewpoint that is adopted in this paper and the method that is used. We have consistently regarded the system of 'Principia' and the generalizations thereof as purely *formal de-*

* H. M. Sheffer, "A Set of Five Independent Postulates for Boolean Algebras, with Applications to Logical Constants," *Trans. Amer. Math. Soc.*, 14 (1913), pp. 481–88.

† J. G. P. Nicod, "A Reduction in the Number of the Primitive Propositions of Logic," *Proc. Camb. Phil. Soc.*, Vol. XIX, Jan., 1917.

velopments,* and so have used whatever instruments of logic or mathematics we found useful for a study of these developments. The fact that one of the interpretations of the system of 'Principia' is part of the informal logic we have used in this study makes the full significance of this *interpretation*, at least with regard to proofs of consistency, uncertain, but it in no way affects the actual content of the paper which is in connection with the *formal systems*.

THE SYSTEM OF PRINCIPIA MATHEMATICA.

1. Description of the System.—Let $p, p_1, p_2, \cdots, q, q_1, q_2, \cdots, r, r_1, r_2, \cdots$ arbitrarily represent the variable elementary propositions mentioned in the introduction. Then by means of the two primitive functions $\sim p$ (read not p—the function of Negation) and $p \vee q$ (p or q—the function of Disjunction) with the aid of the primitive propositions

I. If p is an elementary proposition $\sim p$ is an elementary proposition,
If p and q are elementary propositions $p \vee q$ is an elementary proposition,

we combine these variables to form the various propositions or rather ambiguous values of propositional functions of the system. It is desirable in what follows to have before us the vision of the totality of these functions streaming out from the unmodified variable p through forms of ever growing complexity to form the infinite triangular array

$$\begin{array}{c} p \\ p \vee p, \quad p_1 \vee p_2, \quad \sim p \\ p \vee \sim p, \quad \cdots, \quad \sim p_1 \vee \sim p_2, \quad \cdots, \quad (p_1 \vee p_2) \vee (p_3 \vee p_4), \\ \sim (p_1 \vee p_2), \quad \sim (p \vee p), \quad \sim \sim p \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

and to note and remember that this array of functions formed merely through combining p's by $\sim$'s and $\vee$'s constitutes the entire set of enunciations it is possible to make in the theory of elementary propositions of 'Principia.'

But the actual theory is concerned with the assertion of a certain subset of these functions. We denote the assertion of a function by writing $\vdash$ before it. Then the motive power for the resulting process of deduction is furnished by the two rules of operation:

* For a general statement of this viewpoint see C. I. Lewis, *Loc. Cit.*, Chapter VI, section III.

II. The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $\sim q$, or $(q \vee r)$.*

III. "$\vdash P$" and "$\vdash : \sim P . \vee . Q$" produce "$\vdash Q$."

These enable us to assert new functions from old, or rather in the form in which we have put them, generate new assertions from old. And the complete set of assertions is produced by applying II and III both to the following assertions which give us the start, and to all derived assertions that may result:

$$\text{IV.} \quad \vdash : \sim (p \vee p) . \vee . p, \quad \vdash : \sim [p \vee (q \vee r)] . \vee . q \vee (p \vee r),$$

$$\vdash : \sim q . \vee . p \vee q, \quad \vdash :. \sim (\sim q \vee r) . \vee : \sim (p \vee q) . \vee . p \vee r,$$

$$\vdash : \sim (p \vee q) . \vee . q \vee p.$$

We here again point out what was emphasized in the introduction that this theory concerns itself exclusively with the production of particular assertions through the detailed use of the rules of operation upon the primitive assertions, and as a consequence the set of theorems of this portion of 'Principia' consists of the assertions of a certain number of particular functions of the above infinite set.†

2. Truth-Table Development.‡—Let us denote the truth-value of any proposition p by $+$ if it is true and by $-$ if it is false. This meaning of $+$ and $-$ is convenient to bear in mind as a guide to thought, but in the actual development that follows they are to be considered merely as symbols which we manipulate in a certain way. Then if we attach these two primitive truth-tables to $\sim$ and $\vee$

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* This operation is not explicitly stated in 'Principia' but is pointed out to be necessary by B. Russell in his "Introduction to Mathematical Philosophy," London, 1919, p. 151. Its particular form was suggested to us by the first portion of the operation of "Substitution" given by Lewis, *loc. cit.*, p. 295. It will be noticed that the effect of II is to enable us to substitute any function of the system for a variable of an asserted function.

† We have consistently ignored the idea of definition in this description. We here rigorously follow the authors in saying that definition is a convenience but not a necessity and so need not be considered part of the theoretical development. And so although we too shall at times use its shorthand, we do not encumber our theoretical survey with it.

‡ Truth-values, truth-functions and our primitive truth-tables are described in 'Principia,' Vol. 1, p. 8 and p. 120, but the general notion of truth-table is not introduced. This notion is quite precise with Jevons and Venn (see Lewis, *loc. citus*, p. 74 and pp. 175 et seq. respectively) and has its foundation in the formula for the expansion of logical functions first given by Boole. (G. Boole, "An Investigation of the Laws of Thought," London, Walton, 1854, especially pp. 72–76.) For the relation to Schröder see the footnote to section 3.

p	$\sim p$
$+$	$-$
$-$	$+$

p, q	$p \vee q$
$+\,+$	$+$
$+\,-$	$+$
$-\,+$	$+$
$-\,-$	$-$

we have a means of calculating the truth-values of $\sim p$ and $p \vee q$ from those of their arguments. Now consider any function $f(p_1, p_2, \cdots p_n)$ in our system of functions, which we will designate by F. Then since f is built up of combinations of $\sim$'s and $\vee$'s, if we assign any particular set of truth-values to the p's, successive application of the above two primitive tables will enable us to calculate the corresponding truth-value of f. So corresponding to each of the 2^n possible truth-configurations of the p's a definite truth-value of f is determined. The relation thus effected we shall call the truth-table of f.

For example consider the function

$$\sim (\sim (\sim p \vee q) \vee \sim (\sim q \vee p))$$

which is the ultimate definition of the function $p \equiv q$ of Principia. We have when p is $+$ and q is $+$ the following truth-values of the successive components of the function and so finally of the function:

$$p : +, \quad \sim p : -, \quad \sim p \vee q : +, \quad \sim (\sim p \vee q) : -$$

$$q : +, \quad \sim q : -, \quad \sim q \vee p : +, \quad \sim (\sim q \vee p) : -$$

$$\sim (\sim p \vee q) \vee \sim (\sim q \vee p) : -, \quad \sim (\sim (\sim p \vee q) \vee \sim (\sim q \vee p)) : +$$

the successive truth-values being found by direct application of the primitive tables. In the same way the truth-values for $p+$, $q-$ etc. can be calculated and so we finally get the truth-table of $p \equiv q$, i.e.,

p, q	$p \equiv q$
$+\,+$	$+$
$+\,-$	$-$
$-\,+$	$-$
$-\,-$	$+$

It is needless to say that in actual work this amount of detail is quite unnecessary.

We shall call the number of variables which appear in a function the order of that function as well as that of its truth-table. It is evident that there are 2^{2^n} tables of the nth order. We now prove the

THEOREM. *To every truth-table of whatever order there corresponds at least one function of F which has it for its truth-table.*

For first corresponding to the four tables of the first order $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}+\\-\end{smallmatrix}$, $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}+\\+\end{smallmatrix}$, $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}-\\-\end{smallmatrix}$, $\begin{smallmatrix}+\\-\end{smallmatrix}\big|\begin{smallmatrix}-\\+\end{smallmatrix}$ we have the functions $p \vee p$, $p \vee \sim p$, $\sim (p \vee \sim p)$, $\sim p$. Now assume there is a function for each mth order table. Then in any table of order $m + 1$ the configurations for which p_{m+1} is $+$ constitute an mth order table for which there is some function $f_1(p_1, p_2, \cdots p_m)$. Likewise corresponding to p_{m+1} $-$ we obtain $f_2(p_1, p_2, \cdots p_m)$. Let $p \, . \, q$ stand for $\sim (\sim p \vee \sim q)$ a function which has the truth-table

p, q	$p \cdot q$
$+ \ +$	$+$
$+ \ -$	$-$
$- \ +$	$-$
$- \ -$	$-$

Then it easily follows that the function

$$p_{m+1} \, . f_1(p_1, p_2, \cdots p_m) \, . \vee . \sim p_{m+1} \, . f_2(p_1, p_2, \cdots p_m)$$

has for its truth-table the given $m + 1$st order table.

The functions of F can then be classified according to their tables as follows: those which have all their truth-values $+$, all $-$, or some $+$ and some $-$. We shall call these functions respectively positive, negative, and mixed. This classification is of great importance in connection with the process of substitution which is so fundamental in the postulational development. We shall say that any function obtained from another by the process of substitution is contained in that function. We then have the

THEOREM. *Every function contained in a positive function is positive; every function contained in a negative function is negative; every mixed function contains at least one function for every possible truth-table.*

The first two results are immediate. In the third case note that any mixed function $f(p_1, p_2, \cdots p_n)$ has at least one configuration which yields $+$ and one which yields $-$. Let the truth-value of p_i in the positive configuration be denoted by t_i and in the negative by t'_ι, and construct a function $\phi_\iota(p)$ with the truth-table

p	$\phi_\iota(p)$
$+$	t_ι
$-$	t'_ι

Then $\psi(p) = f(\phi_1(p), \phi_2(p), \cdots \phi_n(p))$ will be $+$ when p is $+$ and $-$ when p is $-$. But by our first theorem there is at least one function $g(q_1, q_2, \cdots q_m)$ corresponding to any table of order m. Hence $\psi[g(q_1, q_2, \cdots q_m)]$ is a function contained in $f(p_1, p_2, \cdots p_n)$ corresponding to that table.

COROLLARY. *Every mixed function contains at least one positive function and one negative function.*

3. The Fundamental Theorem.*—*A necessary and sufficient condition that a function of F be asserted as a result of the postulates II, III, IV is that all its truth-values be* $+$.

Note first that each of the primitive assertions of IV is a positive function. Furthermore from the assertion of positive functions we can only get positive functions. For the only method we have of producing new assertions from old is through the use of II and III. Now II can only produce positive functions since every function contained in a positive function is positive. As for III, if P is $+$ and Q is $-$, $\sim P \vee Q$ is $-$, so that so long as P is a positive function and $\sim P \vee Q$ is a positive function Q must be positive, so that III can only produce positive functions. Hence every asserted function is positive and we have proved the condition necessary.

In order to prove it also sufficient we give a method for deriving the assertion of any positive function. It will simplify the exposition to introduce the other two defined functions of 'Principia' besides $p \,.\, q$ (p and q) given above, viz.,

$$p \supset q \,.=.\, \sim p \vee q \quad Df\dagger; \qquad p \equiv q \,.=.\, p \supset q \,.\, q \supset p \quad Df$$

read "p implies q" and "p is equivalent to q" respectively, and having the tables

p, q	$p \supset q$
$+\,+$	$+$
$+\,-$	$-$
$-\,+$	$+$
$-\,-$	$+$

p, q	$p \equiv q$
$+\,+$	$+$
$+\,-$	$-$
$-\,+$	$-$
$-\,-$	$+$

* The method for testing propositions embodied in this theorem is essentially the same as that given by Schröder for the logical system he has developed. (Ernst Schröder, Vorlesungen über die Algebra Der Logik, Leipzig, Teubner; 2. Bd. 1. Abth, 1891; § 32.) But we believe the range of significance of the proof we have given to be quite different from that of the work of Schröder. For first, as has been emphasized by Lewis (*Loc. cit.*, Chap. IV), formal and informal logic are inextricably bound together in Schröder's development to an extent that prevents the system as a whole from being completely determined. As a result the necessity of the condition of the theorem, which evidently requires such a complete determination if it is to be proved, remains unproved. As for the sufficiency, parts E and C of our proof appear in the proof for the expression of functions given by Schröder. (1. Bd, 1890). Part A, however, seems not to have been given explicitly, while corresponding to part D are all the theoretical difficulties met with in passing from the theory of classes to that of propositions when the development is not strictly formal. Hence the sufficiency of the condition is only incompletely proved. The theorem as given by Schröder is therefore of only partial significance even in his own system; and when transplanted to the system of *Principia* requires independent proof. Finally we may mention that the applications we have made of the theorem depend for their significance on those parts of the proof which do not appear, and could not appear, in Schröder.

† III can now be written "$\vdash P$" and "$\vdash P \supset Q$" produce "$\vdash Q$."

It will be noticed that if we have "$\vdash f_1(p_1, \cdots p_n) \equiv f_2(p_1, \cdots p_n)$" this asserted equivalence must have a positive table by the first part of our theorem, and so f_1 and f_2 must have the same truth-values for the same configurations, i.e., they must have the same truth-table.

The proof is most conveniently given in four stages.

A. We prove the theorem $p \equiv q . \supset . f(p) \equiv f(q)$ where the function f may involve other arguments besides the one indicated and need not involve that. By means of this theorem we shall be able to replace a constituent of a given function by any equivalent function, and have the result equivalent to the given function.

It becomes necessary for the first time to introduce the notion of the rank of a function which we define inductively as follows: the unmodified variable p will be said to be of rank zero, the negative of a function of rank m will be of rank $m + 1$; the logical sum of two functions the rank of one of which equals and the other does not exceed m will be of rank $m + 1$. Each function of F then is of finite rank as well as of finite order.* Returning now to the theorem we notice that it is true for a function of rank zero since it reduces either to $p \equiv q . \supset . p \equiv q$ which follows from $p \supset p$† by II, or to $p \equiv q . \supset . r \equiv r$ which follows from $p \supset . q \supset p$, $r \equiv r$, III and II. Assume now that the theorem holds for functions of rank m and lower. Then it also holds for functions of rank $m + 1$. For if f is of rank $m + 1$ it can be written in the form $\sim f_1(p)$, or, $f_2(p) \vee f_3(p)$ where f_1, f_2 and f_3 are at most of rank m; and then the theorem follows by using $p \equiv q . \supset . \sim p \equiv \sim q$, $p \equiv q . \supset : . r \equiv s : \supset : p \vee r . \equiv . q \vee s$ along with $p \supset q : \supset : q \supset r . \supset . p \supset r$, III and II.

B. Consider now any function $f(p_1, p_2, \cdots p_n)$. Using $\sim (p \vee q) \equiv . \sim p . \sim q$ and $\sim \sim p \equiv p$ with the aid of the equivalence theorem of A and $p \equiv q : \supset : q \equiv r . \supset . p \equiv r$ we finally obtain $f(p_1, p_2, \cdots p_n)$ equivalent to a function $f'(p_1, p_2, \cdots p_n)$ which is expressed merely through combinations of p's and $\sim p$'s by $\cdot$'s and $\vee$'s.

C.‡ If we then apply the distributive law of logical multiplication to f', it will be reduced to an equivalent function consisting of successive logical sums of successive logical products of the p's and $\sim p$'s. If any of these products has neither p_n nor $\sim p_n$ as a factor we can introduce them through the propositions $p \vee \sim p$, and $p : \supset : q . \equiv . p . q$, whence $q : \equiv : (p \vee \sim p) . q : \equiv : p \cdot q . \vee . \sim p . q$. Now apply the commutative and associative laws

* But whereas the number of functions of given order is infinite those of given rank are finite.

† This as well as all other particular assertions that we use without an indication of proof appear in *Principia*, Vol. I, Part *A*.

‡ This portion of the proof is essentially that given by A. N. Whitehead in his "Universal Algebra," p. 46. Camb. Univ. Press, 1898.

of logical multiplication along with $p \,.\, p \,.\equiv.\, p$ so that each product has at most one p_i and one $\sim p_i$. Again using the distributive law for purposes of factorization along with the commutative and associative laws of addition we finally obtain f equivalent to

$$f_1(p_1, p_2, \cdots p_{n-1}) \,.\, p_n \,.\sim p_n : \vee : f_2(p_1, \cdots p_{n-1}) \,.\, p_n \,.\vee.\, f_3(p_1, \cdots p_{n-1}) \,.\sim p_n$$

where one or more of the terms and arguments may not appear.

D. Suppose now that the original function is positive; then this equivalent function will be positive. If in particular it be of first order, it can only be $p \vee \sim p$ or $p \,.\sim.\, p \vee .\, p \vee \sim p$. The first is an asserted function; likewise the second through $p \,.\supset.\, q \vee p$. Hence also $f(p)$ will be asserted through $p \equiv q \,.\supset.\, q \supset p$; and so every positive first order function is asserted. Assume now that this is true for all mth and lower ordered functions and let f be any positive $(m + 1)$st order function. The reduced function being then positive, both f_2 and f_3 will be positive, and hence will be asserted. From the use of $p : \supset : q \,.\supset.\, p \equiv q$, $p \,.\, r \,.\vee.\, p \,.\sim r : \equiv : p \,.\, (r \vee \sim r)$, $p : \supset : S \,.\supset.\, p \,.\, S$, and $p \,.\supset.\, q \vee p$, the reduced function will be asserted and so finally f. Hence every positive function can be asserted and so the proof is complete.

We thus see that given any function the theorem gives a direct method for testing whether that function can or cannot be asserted; and if the test shows that the function can be asserted the above proof will give us an actual *method for immediately writing down a formal derivation of its assertion by means of the postulates of Principia.*

Before we pass on to theorems about the system itself irrespective of truth-tables we give the following definitions which apply directly to the system: a true function is one that can be asserted as a result of the postulates, any other is false; a completely false function is a false function such that every function therein contained is false—otherwise we call it incompletely false. We then have the

COROLLARY. *The set of true, completely false, and incompletely false functions is identical with the set of positive, negative, and mixed functions respectively.*

4. **Consequences of the Fundamental Theorem.**—In the above development the truth-values $+$, $-$ were arbitrary symbols which were found related in certain suggestive ways through the fundamental theorem. We are now in a psoition to give direct definitions of these truth-values in terms of the postulational development. In fact we shall define $+$ to be the set of true functions, $-$ the set of completely false functions. The truth-value of a function will then exist when and only when it is true or completely false, and it will be defined as that class $(+, -)$ of which it is a member. The content of the fundamental theorem consists now of these two theorems:

1. The truth-value of $\sim p$ and $q \vee r$ exists whenever the truth-values of p, q and r exist, and depends only on those truth-values as given by the primitive tables. It therefore follows that the same is true of any function of F, and that the truth-table of such a function can be directly calculated from the primitive tables.

2. The fundamental theorem as stated, or else in the form: if f_1 and f_2 is any pair of positive and negative functions respectively, then a necessary and sufficient condition that a function $f(p_1, p_2, \cdots p_n)$ be asserted is that each of the 2^n contained functions found by substituting f_1 and f_2 for the p's is asserted. It will be noticed that theorem (1) tells us how to determine whether these latter are asserted.

We now pass on to several theorems about the system.

THEOREM. *It is possible to find 2^{2^n} functions of order n such that no two of them are equivalent and such that every other function of order n is equivalent to one of these.*

For we can find 2^{2^n} functions corresponding to the 2^{2^n} different tables of order n. The equivalence of any two of these will then not have a positive table and so will not be asserted. On the other hand any other nth order function will have the same table as one of the 2^{2^n} possible tables, and so the corresponding equivalence will be positive and hence asserted.

THEOREM. *An incompletely false function contains at least one function for each given function which is equivalent to that given function.*

COROLLARY. *An incompletely false function contains at least one true function and one completely false function.*

THEOREM. *The negative of a completely false function is true.*

For a completely false function has a negative truth-table, and so its negative will have a positive table and hence be asserted. It is worth noticing that although this theorem is immediate once we have the fundamental theorem it would be quite difficult without it.

COROLLARY. *Every function of F is either true, or its negative is true, or it contains both a true function and one whose negative is true.*

THEOREM. *The system of elementary propositions of 'Principia' is consistent.*

For if it were inconsistent we would have both a function and its negative asserted. But then both the function and its negative would have to have positive tables whereas if a function has a positive table its negative has a negative table.*

THEOREM. *Every function of the system can either be asserted by means of the postulates or else is inconsistent with them.*

* This argument requires merely the first part of the fundamental theorem which was proved quite simply.

For if a function be not asserted as a result of the postulates it will contain a function whose negative can be so asserted. If then we assert the original function, the contained function will be asserted so that we have asserted both a function and its negative, i.e., we have a contradiction.

COROLLARY. *A function is either asserted as a result of the postulates or else its assertion will bring about the assertion of every possible elementary proposition.*

For by the theorem we would obtain the assertion of both a function and its negative and so by $\sim p . \supset . p \supset q$ the assertion of the unmodified variable q. But q then represents any elementary proposition.

In conclusion let us note that while the fundamental theorem shows that the postulates bring about the assertion of those and only those theorems which should belong to the system, this last theorem enables us to say that they also automatically exclude the very possibility of any added assertions.

GENERALIZATION BY TRUTH-TABLES.

5. General Survey of the Systems Generated.—The system we have studied in the preceding sections is a particular system depending upon the two primitive functions $\sim p$ and $p \vee q$. Two modes of attack have presented themselves. On the one hand we have the original postulational method, on the other the truth-table development. In passing to a general study of systems of the kind discussed these two methods present themselves as instruments of generalization. We reserve the postulational generalization for the next portion of our paper and now take up the truth-table generalization.

To gain complete generality let us assume for our primitives μ arbitrary functions with an arbitrary number of arguments which we will designate by

$$f_1(p_1, p_2, \cdots p_{m_1}), f_2(p_1, p_2, \cdots p_{m_2}), \cdots f_\mu(p_1, p_2, \cdots p_{m_\mu})$$

and let us attach an arbitrary truth-table to each. By successive combinations of these functions with different or repeated arguments we generate the set of derived functions which as before we designate by F. Again each function of F will possess a truth-table in virtue of the tables of the primitive functions of which each is composed. Denote the set of truth-tables thus generated by T. Then whereas in the system of 'Principia' T consists of all possible truth-tables, this will not necessarily be the case here.

In another paper we completely determine all the possible systems T and show that there are 66 *systems that can be generated by tables of third and lower order, and* 8 *infinite families of systems that are generated by the introduction of fourth and higher ordered tables.*

If two systems have the same truth-tables the primitives of each can evidently be expressed in terms of those of the other so that truth-tables are preserved. We can then say that each system has a representation in the other and the two are equivalent. In particular *every truth-system has a representation in the system of Principia* while *every complete system,* i.e., having all possible truth-tables, *is equivalent to it.* In the aforementioned paper we also determine the ways in which a complete system may be generated, and it turns out that one table alone is sufficient to generate it, and it can be either of these two

$$\begin{array}{cc|c} + & + & - \\ + & - & + \\ - & + & + \\ - & - & + \end{array} \qquad \begin{array}{cc|c} + & + & - \\ + & - & - \\ - & + & - \\ - & - & + \end{array}$$

a result first given by Sheffer as stated in the introduction.

The truth-table development for complete systems is essentially the same as that given in section 2. It is easy to prove for all systems the

THEOREM. *Every function contained in a positive function is positive; every function contained in a negative function is negative; every mixed function contains a function for every table of the system.*

6. Postulates for a Complete System.—We now show how to construct a set of postulates for any complete system such that: *the set of asserted functions is identical with the set of positive functions, while the assertion of any other function brings about the assertion of every elementary proposition* a property which also characterized the system of 'Principia.'

Let $\sim' p$ and $p \vee' q$ be functions in the given complete system with the tables of $\sim$ and $\vee$. Out of $\sim'$ and $\vee'$ we then construct $p \supset' q$ and $p \equiv' q$ as $p \supset q$ and $p \equiv q$ are found from $\sim$ and $\vee$, and also $f_1'(p_1, \cdots p_{m_1})$, $\cdots, f_\mu'(p_1, \cdots p_{m_\mu})$ with the same tables as $f_1(p, \cdots p_{m_1}), \cdots, f_\mu(p_1, \cdots p_{m_\mu})$. This is possible since $\sim$ and $\vee$, and so $\sim'$ and $\vee'$ can generate a complete system. All the functions $\sim'$, $\vee'$, $\supset'$, $\equiv'$, f_1', $\cdots$ f_μ' are ultimately expressed in terms of the f's and so belong to the system. Construct now the following set of postulates:

I. If $p_1, \cdots p_{m_1}$ are elementary propositions, $f_1(p_1, \cdots p_{m_1})$ is.

. . . .

If $p_1, \cdots p_{m_\mu}$ are elementary propositions, $f_\mu(p_1, \cdots p_{m_\mu})$ is.

II. The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $f_1(q_1, \cdots q_{m_1})$, $\cdots$ or $f_\mu(q_1, \cdots q_{m_\mu})$.

III. "$\vdash P$" and "$\vdash P \supset' Q$" produces "$\vdash Q$."

IV. (1) $\vdash : p \vee' p . \supset' p$ (a) $\vdash . f_1(p_1, p_2, \cdots p_{m_1}) \equiv' f_1'(p_1, p_2, \cdots p_{m_1})$,

. . . .

(5) $\vdash \cdots$ (u) $\vdash . f_\mu(p_1, p_2, \cdots p_{m_\mu}) \equiv' f_\mu'(p_1, p_2, \cdots p_{m_\mu})$.

where (1)–(5) are the assertions of IV in sec. 1 with $\sim'$ and $\vee'$ in place of $\sim$ and $\vee$.

That all asserted functions are positive can be verified as in the proof of sec. 4. As for the converse, note that III and IV (1)–(5) being of the same form as III and IV of sec. 4 will yield the assertion of all positive functions expressed in terms of $\sim'$ and $\vee'$. By the use of (a)–(u) every function can be shown to be equivalent ($\equiv'$) to some function expressed by $\sim'$ and $\vee'$ and so every positive function will be asserted. In the same way the assertion of any non-positive function will bring about the assertion of a non-positive function in $\sim'$ and $\vee'$ alone, and so of any proposition.

We thus see that complete systems are equivalent to the system of 'Principia' not only in the truth table development but also postulationally. As other systems are in a sense degenerate forms of complete systems we can conclude that no new logical systems are introduced.

7. Application to Nicod's Postulate Set.—Although, as in most existence theorems, the above set of postulates may not be the simplest in any one case, it can be used to advantage in showing that a given set has the same property as it possesses. For this purpose we show directly that all asserted functions are positive, and then that by means of the given postulates (a) each of our formal postulates may be derived (b) that the results derivable by our informal postulates can also be derived by the given ones.*

As an example we consider the set of postulates given by Nicod for the theory of elementary propositions in terms of the single primitive function of Sheffer's which Nicod denotes by $p|q$ and is termed incompatibility by Russell.† It is the first of the two functions given in section 5 as generating a complete system. Nicod gives the definitions

$$\sim p \,.=.\, p|p \quad Df, \qquad p \vee q \,.=.\, p/p|q/q \quad Df$$

which we take to be our $\sim' p$ and $p \vee' q$ respectively. His $p \supset q \,.=.\, p|q/q Df$ however is not our $p \supset' q$ which is $\sim' p \vee' q$. The primary distinction of his system is that he uses but one formal primitive proposition.

In carrying out the proof suggested we merely note that by means of his informal proposition "$\vdash P$" and "$\vdash P|R/Q$" produce "$\vdash Q$" we get the effect of "$\vdash P$" and "$\vdash P|Q/Q$" i.e., "$\vdash P \supset Q$" produce "$\vdash Q$" when $R = Q$. Since he has $p \supset' q \,.\supset.\, p \supset q$ we thus get the effect of "$\vdash P$"

* That the informal postulates of a system must be proved effectively replaced by others in another system is a precaution rarely taken in discussions of equivalence or dependence of logical systems. Such a discussion is unnecessary in ordinary mathematical systems since their distinctive postulates are all formal, the informal ones being those of a common logic. But in comparing logical systems, which usually do contain different informal postulates, such a discussion is fundamental.

† B. Russell, *loc. cit.*, chap. XIV.

and "$\vdash P \supset' Q$" produce "$\vdash Q$" our III. Likewise each function IV is proved with however $\supset$ in place of $\supset'$. But by means of $p \supset q . \supset . p \supset' q$ this too is remedied. We then easily complete the proof of the

THEOREM. *If in Nicod's system we give to $p|q$ the table*

p, q	$p\|q$
$+ +$	$-$
$+ -$	$+$
$- +$	$+$
$- -$	$+$

then the set of asserted functions is identical with the resulting set of positive functions; and the assertion of any other function would bring about the assertion of every elementary proposition.

GENERALIZATION BY POSTULATION.

8. The Generalized Set of Postulates.—As in the truth-table development we assume arbitrary primitive functions of propositions

$$f_1(p_1, p_2, \cdots p_{m_1}), \cdots, f_\mu(p_1, p_2, \cdots p_{m_\mu});$$

but in place of the arbitrary associated truth-tables we have a set of postulates of the following form. We have tried to preserve all the informal properties of the postulates of 'Principia' (and of sec. 5) but generalize the formal properties completely.

I. (As in sec. 5.)
II. (As in sec. 5.)

$$\begin{array}{llll}
\text{III.} & \text{“}\vdash g_{11}(P_1, P_2, \cdots P_{k_1})\text{”} & \cdots & \text{“}\vdash g_{\kappa 1}(P_1, P_2, \cdots P_{k_\kappa})\text{”} \\
 & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\
 & \text{“}\vdash g_{1\kappa_1}(P, P_2 \cdots P_{k_1})\text{”} & \cdots & \text{“}\vdash g_{\kappa\kappa_\kappa}(P_1, P_1, \cdots P_{k_\kappa})\text{”} \\
 & \text{produce} & \cdots & \text{produce} \\
 & \text{“}\vdash g_1(P_1, P_2, \cdots P_{k_1})\text{”} & \cdots & \text{“}\vdash g_\kappa(P_1, P_2, \cdots P_{k_\kappa})\text{”}
\end{array}$$

where the P's are any combinations of f's including the special case of the unmodified variable, while the g's are particular combinations of this kind which need not have all the indicated arguments.

$$\begin{array}{ll}
\text{IV.} & \vdash h_1(p_1, p_2, \cdots p_{l_1}) \\
 & \vdash h_2(p_1, p_2, \cdots p_{l_2}) \\
 & \cdots\cdots \\
 & \vdash h_\lambda(p_1, p_2, \cdots p_{l_\lambda})
\end{array}$$

where the h's are particular combinations of the f's.

The retention of I and II which are characteristic of the theory of

elementary propositions is our justification for giving that name to the systems that may be generated by the above set of postulates. In what follows we give what we consider to be merely an introduction to the general theory.

9. Definition of Consistency and Related Concepts.—The prime requisite of a set of postulates is that it be consistent. Since the ordinary notion of consistency involves that of contradiction which again involves negation, and since this function does not appear in general as a primitive in the above system a new definition must be given.

Now an inconsistent system in the ordinary sense will involve the assertion of a pair of contradictory propositions which as we have seen will bring about the assertion of every elementary proposition through the assertion of the unmodified variable p. Conversely since p stands for any elementary proposition its assertion would yield the assertion of contradictory propositions and so render the system inconsistent. The two notions are thus equivalent in ordinary systems; and since one retains significance in the general case we are led to the

DEFINITION.—*A system will be said to be inconsistent if it yields the assertion of the unmodified variable p.*

In a consistent system we may then define a true function as one that can be asserted as a result of the postulates. Instead of defining a false function as one not true, we give the following

DEFINITION. *A false function is one such that if its assertion be added to the postulates the system is rendered inconsistent.*

We can then state that in the system of 'Principia' every function is true or false. This suggests the

DEFINITION. *If every function of a consistent system is true or false the system will be said to be closed.**

As a justification of this name we may note that the postulates of such a system automatically exclude the possibility of any added assertions—a state of affairs we believe to be highly desirable in the final form of a logical theory.

10. Properties of Consistent Systems.—In all that follows we assume that the system discussed is consistent. If it be inconsistent one could hardly say anything more about it.

We turn to a theorem which will give us most of the results of this section. But first we must state two lemmas which we do not further prove.

LEMMA 1.—If a given set of functions gives rise to some other function in accordance with II and III, and if these functions involve certain letters

* Had the name not been in use in a different connection we should have introduced the term categorical.

$r_1, r_2, \cdots r_i$ upon which no substitution is made in the process, then the same deductive process will be valid if we have given the original functions with an arbitrary substitution of the r's as described in II provided this substitution is also made throughout the process.

LEMMA 2.—The most general process of obtaining an assertion from a given set of assertions in accordance with II and III can be reduced to first asserting a number of functions in accordance with II, and then applying II and III in such a way that no substitutions are made on the arguments of those functions.

THEOREM. *Every false function contains a finite set of untrue first order functions $\phi_1(p), \phi_2(p), \cdots \phi_\nu(p)$ such that whenever p is replaced by an untrue function at least one of these functions remains untrue.*

By the definition of false functions there must be some deductive process whereby from the given false function and true functions we assert p. By lemma 2 we can replace this process by another where from the given false function and true functions we obtain certain contained functions from which without substitution of the arguments we obtain p. Now first by lemma 1 we can equate to p all the arguments thus appearing and still have a valid deductive process for obtaining p. Denote the resulting untrue functions which are contained in the original false function by $\phi_1(p)$, $\phi_2(p), \cdots \phi_\nu(p)$. Then secondly by lemma 1 we can replace p by any function ψ and still have a valid process which now consists in obtaining ψ from certain true functions and $\phi_1(\psi), \cdots \phi_\nu(\psi)$. If then each $\phi_\iota(\psi)$ were true, ψ, being obtained from true functions in accordance with II and III would be true. It follows that if ψ be untrue, some $\phi_\iota(\psi)$ must be untrue.

THEOREM. *Every false function contains an infinite number of untrue first order functions; and if the system has at least one false function of order greater than one, then each false function contains an infinite number of untrue functions of every order.*

By the above theorem the false function contains at least one untrue function $\phi_{\iota_1}(p)$. By the same theorem some $\phi_{\iota_2}\phi_{\iota_1}(p)$ must be untrue, etc., through $\phi_{\iota_j}, \phi_{\iota_{j-1}} \cdots \phi_{\iota_1}(p)$. These are all different being of different rank, and are all contained in the given function.

The last part of the theorem may then be proved by showing that by replacing equal by unequal variables in the infinity of functions thus gotten from the false function of order greater than one we get untrue functions of every order, and so by the above method an infinite number of every order in every false function.

We have immediately the

THEOREM. *A necessary and sufficient condition that a function of a closed system be true is that all contained first order functions be true.*

COROLLARY. *It is also necessary and sufficient that all those of rank greater than some finite integer ρ be true.*

In analogy with corresponding ideas in the system of 'Principia' define a completely untrue function as one in which all contained functions are untrue with a similar definition for completely false. We then have the interesting

THEOREM. *If a system has a completely untrue function, then every false function contains a completely untrue function.*

Every function contained in the completely untrue function makes at least one $\phi_\iota(p)$ of a false function untrue. If ψ is such a contained function which makes say $\phi_{\iota_1}(p)$ true, then ψ will be completely untrue, and all contained functions will make $\phi_{\iota_1}(p)$ true yet some remaining $\phi_\iota(p)$ untrue. By repeating this process we finally obtain a function ψ' such that all contained functions make each $\phi_\iota(p)$ of a set that remains untrue. Each such $\phi_\iota(\psi')$ will then be a completely untrue function in the given one.

COROLLARY. *If a closed system has a completely false function every false function contains a completely false function.*

If we call such a system completely closed we have the stronger

THEOREM. *In a completely closed system every false function $f(p_1, p_2, \cdots, p_n)$ contains a completely false function $f(\psi_1(p), \psi_2(p), \cdots, \psi_n(p))$ where each $\psi_\iota(p)$ is either true or completely false.*

By equating all variables to p in the function of the corollary we get such a completely false function where some ψ's may be incompletely false. These are then eliminated by successively substituting for p functions which make them true.

COROLLARY. *A necessary and sufficient condition that a function of a completely closed system be true is that all contained first order functions found by substituting true or completely false functions for the arguments be true.*

This property begins to approximate to the truth-table method. It leads us easily to the following criterion for a completely closed postulational system being a truth-system which we state without proof.

THEOREM. *A necessary and sufficient condition that a completely closed postulational system be a truth-system is that a true first order function remains true whenever we replace a true or completely false constituent function by any other true or completely false first order function respectively.**

* In making a more complete study of the postulational generalization it would be desirable to classify all the systems that may result more or less in the way in which we have classified truth-systems through the associated systems of truth-tables. In this connection we might define the order of a set of postulates as the largest number of premises used in deriving a conclusion in III, and the order of a system as the lowest order a set of postulates deriving it can have. It is then of interest to note that *whereas the set of postulates of the system of 'Principia' is of the second order, the system itself is of the first order.*

m-VALUED TRUTH-SYSTEMS.*

11. The Generalized ($\sim$, $\vee$) System.—We have seen that the truth-table generalization, at least with regard to complete systems, is included in the postulational development. We now show that the latter is more general by presenting a new class of systems, distinct from the two-valued systems of symbolic logic, which can be generated by a completely closed set of postulates.

In these systems instead of the two truth-values $+$, $-$ we have m distinct "truth-values" $t_1, t_2, \cdots, t_m$ where m is any positive integer. A function of order n will now have m^n configurations in its truth-table, so that there will be m^{m^n} truth-tables of order n. Calling a system having all possible tables complete, we now show that the following two tables generate a complete system.

p	$\sim_m p$
t_1	t_2
t_2	t_3
$\cdots$	$\cdots$
t_m	t_1

p, q	$p \vee_m q$	
$t_1 t_1$	t_1	
$\cdots$	$\cdots$	
$t_{\iota_1} t_{j_1}$	t_{ι_1}	$\iota_1 \leqq j_1$
$\cdots$	$\cdots$	$\iota_2 \geqq j_2$
$t_{\iota_2} t_{j_2}$	t_{j_2}	
$\cdots$	$\cdots$	
$t_m t_n$	t_m	

We see that $\sim_m p$, the generalization of $\sim p$, permutes the truth-values cyclically, while $p \vee_m q$, the generalization of $p \vee q$ has the higher of the two truth-values.†

To construct a function for any first order table, of which there are m^m, note that

$$t_1(p) \,.=.\, p \vee \sim_m p \,.\, \vee_m \sim_m^2 p : \vee_m \cdots \sim_m^{m-1} p \quad Df,$$

where $\sim^2 p \,.=.\, \sim\sim p \quad Df$, etc., has all its truth values t_1. Then

$$\tau_{m_1}(p) \,.=.\, \sim_m^{m-1}(\sim_m^{m-1}(\sim_m t_1(p) \,.\vee_m .\, p) : \vee_m \,.\, \sim_m^{m_1} p) \quad Df$$

has all values t_m except the first which is t_{m_1}. Any first order table

p	$f(p)$
t_1	t_{m_1}
t_2	t_{m_2}
$\cdots$	$\cdots$
t_m	t_{m_m}

can then be constructed by the function

* See Lewis, *loc. cit.*, p. 222 for the term "Two-Valued Algebra."

† The higher truth-value has here the smaller subscript.

$$\tau_{m_1}(p) \,.\, \vee_m \,.\, \tau_{m_2}(\sim_m^{m-1} p) : \vee_m \,.\, \tau_{m_3}(\sim_m^{m-2} p) :. \vee_m :\cdots \tau_{m_m}(\sim_m p).$$

Construct now a function for the table

p	$\overline{\sim}_m p$
t_1	t_m
t_2	t_{m-1}
$\cdots$	$\cdots$
t_m	t_1

and define $p \,._m q \,.=.\, \overline{\sim}_m(\overline{\sim}_m p \,.\, \vee_m \,.\, \overline{\sim}_m q)$ *Df* which is the generalization of $p\,.\,q$ and has the lower of the two truth values of its arguments. We can now construct a table all of whose values are t_m except for one configuration $t_{m_1}, t_{m_2}, \cdots, t_{m_n}$ when it is $t_{m_{m_1 m_2 \cdots m_n}} = t_\mu$ by the function

$$\tau_\mu(\sim_m^{m-m_1+1} p_1) \,._m \tau_\mu(\sim_m^{m-m_2+1} p_2) \,._m \cdots \tau_\mu(\sim_m^{m-m_n+1} p_n),$$

and so any table by constructing such a function for each configuration and then "summing up" by $\vee_m$.

12. Classification of Functions—the m dimensional Space Analogy.— The generalization of the classification of functions into positive, negative and mixed is afforded us by the following

THEOREM. *A function contains at least one function for every truth-table whose values are contained among the values of the given table.*

Let $t_{m_1} \cdots t_{m_\mu}$ be the truth-values that appear in the table of a given function $f(p_1, p_2, \cdots p_n)$. Then we can pick out μ configurations having these values respectively. Construct functions $\phi_\iota(p)$ such that when p has the vlaue t_{m_j} of one of these configurations, $\phi_\iota(p)$ have the value of p_i in that configuration. It is then easily seen that $f(\phi_1(p), \cdots, \phi_n(p))$ has the value t_{m_j} whenever p has the value t_{m_j}. If then $\psi(q_1, q_2, \cdots, q_l)$ have a table whose values are among the t_{m_j}'s, $f(\phi_1(\psi), \cdots, \phi_n(\psi))$ will be a function contained in the given function with that table.

We are thus led to a classification of functions by means of their truth-tables such that the set of tables of contained in a given function is the same for all functions in a given class. We then have m classes of functions where but one truth-value appears, $[m(m-1)]/2!$ with two truth-values, $\cdots$, $[m(m-1)\cdots(m-\mu+1)]/\mu!$ with μ truth-values, $\cdots$, one class with all m truth-values. We thus have $2^m - 1$ classes of functions which when $m = 2$ reduces to the three classes of positive, negative and mixed functions.

These formulæ suggest an analogy which, if well founded, is of great interest. For this purpose replace the set of functions having all of a given set of μ truth-values by all functions whose values are among these μ values. If then we compare the functions of our complete system to the points of a

space of m dimensions,* the m classes of functions with but one truth-value would correspond to the m coordinate axes, the $[m(m-1)]/2!$ classes of functions with no more than two truth-values to the $[m(m-1)]/2!$ coordinate planes, etc., so that except for the absence of an origin all properties of determination and intersection within the coordinate configurations go over. If then we attach the name m-dimensional truth-space to our system, we observe the following difference, that whereas the highest dimensioned intuitional point space is three, the highest dimensioned intuitional proposition space is two. But just as we can interpret the higher dimensioned spaces of geometry intuitionally by using some other element than point, so we shall later interpret the higher dimensioned spaces of our logic by taking some other element than proposition.

13. Truth-Table Characteristics of Asserted Functions.—The following analysis presupposes that in constructing a set of postulates for the system we at least wish to impose the

CONDITION.—*If a function is asserted, all functions with the same truth-table will be asserted.*

It follows from the theorem of the preceding section that under the given condition, *if a function is asserted, every function of the truth-space it determines is asserted.*

We can now prove that *if the system is to be completely closed its asserted functions must constitute a single truth-space contained in the given truth space.* For if there were at least two such spaces, then a function having all their truth-values would be false, and so would contain a completely false function. This in turn would contain functions with but one truth-value; and these being therefore in one of the two given spaces would be true which contradicts their being in a completely false function.

No loss of generality ensues if we take the truth values of this contained truth-space of asserted functions to be $t_1, t_2, \cdots t_\mu$, where, to avoid degenerate cases $0 < \mu < m$. We now show that a completely closed set of postulates can be constructed for all such systems.

14. A Completely Closed Set of Postulates for the Systems.—I and II are determined directly as in the general case. To obtain III, construct a function $p \supset^{\mu}_{m} q$ whose table is given by the following: when the truth-value of p is that of q or lower, $p \supset^{\mu}_{m} q$ will have the value t_1, while if the truth-value of p is above that of q, then if the value of p is t_μ or higher, $p \supset^{\mu}_{m} q$ will have the value of q, while if it is below t_μ, say t_ν and that of q is $t_{\nu'}$, then the truth-value of $p \supset^{\mu}_{m} q$ will be $t_{\nu'-\nu+1}$. III will then be simply

* Or we might take the truth-table as element in which case the system is perhaps smoother than before.

$$\text{“}\vdash P\text{”}$$
$$\text{“}\vdash P \supset^{\mu}_{m} Q\text{”}$$
produce
$$\text{“}\vdash Q\text{”}$$

Now by generalizing each part A, B, C, D of the proof of the fundamental theorem of sec. 3 it can be shown that by the assertion of a finite number of functions with values from t_1 to t_μ all such can be obtained.* If then we assert these functions in IV we shall have every function in the μ-space asserted. Furthermore no others can be asserted for by the use of II and III we can only get functions with values from t_1 to t_μ by means of functions similarly restricted. This is obvious in II while in III if the value of P is from t_1 to t_μ while that of Q is below t_μ, then from the above definition of the table of $P \supset^{\mu}_{m} Q$ it would have the value of Q and so be below t_μ. But that contradicts the assumption that the premises had values from t_1 to t_μ.

This set of postulates will then give the proper set of true functions. Furthermore let us suppose that we assert a function with at least one value below t_μ. This will contain a function $\phi(p)$ with but one value, and that below t_μ. By II, $\phi(p)$ will be asserted. Furthermore since $\phi(p) \,.\supset^{\mu}_{m}.\, \phi(p) \supset^{\mu}_{m} \sim_m \phi(p)$ has its value t_1 it will be asserted, and so we obtain by III $\sim_m \phi(p)$. Repetition of this process will finally give us a function $\psi(p)$ with but one value t_m. But $\psi(p) \cdot \supset^{\mu}_{m} p$ is asserted having but one value t_1. We thus obtain the assertion of p. The system is therefore closed. And since all functions with values from $t_{\mu+1}$ to t_m are completely false, the system is completely closed.

15. Comparison of Systems.—As in the truth-table development we can generalize the systems by using arbitrary functions as primitives, and as was done there we can show how to generate a complete m-dimensioned system by one second order function, and how to give a completely closed set of postulates for all complete systems. The problem of determining all possible systems of m-dimensional truth-tables, however, is one we have not considered, though its solution would through considerable light on the ordinary problem.

We turn now to the following

DEFINITIONS. *A closed system S with primitives $f_1, f_2, \cdots f_n$ has a representation in a closed system S' with primitives $f'_1, f'_2, \cdots f'_{n'}$ if we can so replace the f's by functions in S' that a function in S will be true when and only when the correspondent in S' is true.*

Two systems are equivalent if each has a representation in the other.

Denote a complete m-dimensional truth-system with the asserted functions forming a truth-space of μ dimensions by ${}_{\mu}T_m$. We then have the

* Lack of space prevents us from giving the details.

THEOREM. *Two complete truth-systems* ${}_{\mu}T_m$ *and* ${}_{\mu'}T'_{m'}$ *are equivalent when and only when* $\mu = \mu'$ *and* $m = m'$.

The conditions are clearly sufficient since we can make truth-values correspond. To prove them necessary suppose $m > m'$. If we construct m^m functions of first order in T with different truth-tables then there will be two, $\phi_1(p)$, $\phi_2(p)$ whose correspondents $\phi'_1(p)$, $\phi'_2(p)$ have the same truth-tables since there are in T' only $m'^{m'}$ of first order. Let $\chi(p, q)$ have value t_1 when p and q have the same value and t_m otherwise. Then $\chi(\phi_1, \phi_1)$ is true; hence $\chi'(\phi'_1, \phi'_1)$ is. ϕ'_2 having the same table as ϕ'_1, $\chi'(\phi'_1, \phi'_2)$ is true, and hence $\chi(\phi_1, \phi_2)$ the correspondent. But that would make ϕ_1 have the same table as ϕ_2. Now suppose $\mu > \mu'$. If ϕ have all the values from t_1 to t_μ and no others there are μ^μ functions with values t_1 to t_μ of the form $\psi\phi(p)$. These will then be asserted and so the correspondents will be asserted and have values t'_1 to $t'_{\mu'}$. Since we can only have $\mu'^{\mu'}$ functions $\psi'\phi'(p)$ with different tables, we can find two of the μ^μ correspondents with the same table. The above contradiction then results as before.

For representation we have only found the

THEOREM. *To represent* ${}_{\mu}T_m$ in ${}_{\mu'}T'_{m'}$, *it is necessary to have* $\mu \leq \mu'$, $m \leq m'$; *it is sufficient to have* $\mu \leq \mu'$, $m - \mu \leq m' - \mu'$.

COROLLARY. *A necessary and sufficient condition that* ${}_{\mu}T_m$ *have a representation in* ${}_{\mu}T'_{m'}$, *is that* $m \leq m'$.

It is of interest to note as a result that the only complete truth-systems equivalent to the system of 'Principia' are ${}_1T_2$'s; and though it can be represented in every complete truth-system, only ${}_1T_2$'s can be represented in it. We have thus verified our statement that we obtain essentially new logical systems.

16. Interpretation of m-valued Truth-systems in Terms of Ordinary Logic.—Let the elementary proposition of the $(\sim_m, \vee_m)$ system be interpreted as an ordered set of $(m-1)$ elementary propositions of ordinary logic $P = (p_1, p_2, \cdots p_{m-1})$ such that if one proposition is true all those that follow are true. P will be then be said to have the truth-value t_1 if all the p's are true, t_2 if all but one are true, etc. Also P will be said to be true if at most $(\mu - 1)p$'s are false.

If $P = (p_1, p_2, \cdots p_{m-1})$, $Q = (q_1, q_2, \cdots q_{m-1})$ we define

$$P \vee_m Q \,.=.\, (p_1 \vee q_1, p_2 \vee q_2, \cdots p_m \vee q_m) \quad Df$$

$$\sim_m P \,.=.\, (\sim(p_1 \vee p_2 \vee \cdots p_{m-1}), \sim(p_1 \vee \cdots p_{m-1}) \,.\vee.\, p_1 \,.\, p_2, \cdots,$$
$$\sim(p_1 \vee \cdots p_{m-1}) \,.\vee.\, p_{m-2} \,.\, p_{m-1}) \quad Df$$

We easily justify these definitions by showing first that $P \vee_m Q$ and $\sim_m P$ are "elementary propositions" when P and Q are, and secondly that they

have the proper truth-tables. Thus in $P \vee_m Q$ the first $p_\iota \vee q_\iota$ to be true is the first for which either p or q is true; also all later terms will have p or q true and so will be true. $P \vee_m Q$ is therefore elementary and has the required table.

But in spite of this representation ${}_1T_2$ still appears to be the fundamental system since its truth-values correspond entirely to the significance of true and completely false, whereas in ${}_\mu T_m$, $m > 2$ either $\mu > 1$ or $m - \mu > 1$, and this equivalence no longer holds. We must however take into account the fact that our development has been given in the language of ${}_1T_2$ and for that very reason every other kind of system appears distorted. This suggests that *if* we translate the entire development into the language of any one ${}_\mu T_m$ by means of its interpretation, then it would be the formal system most in harmony with regard to the two developments.

NOTE ON SCHLÄFLI'S ELLIPTIC MODULAR EQUATIONS.

By Arthur Berry.

In a former paper in this journal* I proved some properties of the elliptic modular equations substantially in the form commonly known as Schläfli's equations. I worked with a modular function $x(\tau) = 2^{-1/6}\chi(\tau)$, where χ is Hermite's function† and showed that, for transformations of prime order n (> 3), (1) when n is of the form $4p - 1$, corresponding to $\tau = i$, $x = 2^{-1/4}$ the roots of the modular equations are equal in pairs and branched, each pair corresponding to a branch point of order 1 on the corresponding Riemann surface and that (2) when n is of form $4p + 1$, there are $n - 1$ roots equal in pairs and branched, and two isolated roots. I showed further that the two isolated roots are always of the form $\epsilon^\lambda x(i)$ (λ an integer, $\epsilon = e^{2i\pi/24}$), and that for $n = 8p - 3$, the two values of ϵ^λ are $\pm i$ ($\lambda = \pm 6$ *mod.* 24), while for $n = 8p + 1$ both values are -1, or both values 1, but was unable to give any simple criterion distinguishing these last two cases.

The object of this note is to establish these results as to the isolated roots in a somewhat simpler way, avoiding the rather troublesome quadratic transformation used before, and to distinguish between the last two cases.

It is known that for a modular substitution $\{(c + d\tau)/(a + b\tau), \tau\}$ of Hermite's first type (a, d odd, b, c even)

$$x\{(c + d\tau)/(a + b\tau)\} = \epsilon^\lambda x(\tau),$$

where $\lambda = \frac{1}{2}(b - c)(bcd - a)$;‡ a similar equation holds for substitutions of the second type, but as these can be derived from those of the first type by applying the substitution $T(T\tau = -1/\tau)$ and we are only concerned with $\tau = i$, so that $T\tau = \tau$, it is enough to consider substitutions of the first type. Hence in order to prove that $x\{(48r + i)/n\} = \epsilon^\lambda x(i)$, it is enough to prove that it is possible to find integers a, b, c, d, where a, d are odd, b, c even and $ad - bc = 1$, and an integer r such that

$$(48r + i)/n = (c + di)/(a + bi). \tag{1}$$

If n is a prime number of the form $4p + 1$, it is a well known result that

* On Elliptic Modular Equations for Transformations of Orders 29, 31, 37; Vol. XXX, pp. 156–169.

† Sur la résolution de l'équation du quatrième degré; *Comptes Rendus*, Vol. 46 (1858), *Oeuvres*, Vol. II, p. 28.

‡ Tannery and Molk, *Fonctions Elliptiques*, Vol. II, Table XLVI.

we can choose a and b (a odd, b even) so that $a^2 + b^2 = n$.* With this choice of a and b (1) is equivalent to

$$ac + bd = 48r, \tag{2}$$

$$ad - bc = 1. \tag{3}$$

We can now choose c, d to satisfy (3) and can further arrange so that c is even d odd; if c', d' is one such solution, the general solution is $c = c' + 2ka$, $d = d' + 2kb$ (k integral), and (2) can be satisfied, if

$$2k(a^2 + b^2) + ac' + bd' \equiv 0, \quad mod\ 48,$$

or

$$kn \equiv -\tfrac{1}{2}(ac' + bd'), \quad mod\ 24;$$

since n is prime and $ac' + bd'$ is even, this congruence can be satisfied; a, b, c, d are now found, and r is given by (2).

It remains to determine the integer λ (*mod.* 24), which depends on the congruences *mod.* 3 and *mod.* 16, satisfied by a, b, c, d.

From (2) and (3), it follows at once that if any one of $a, b, c, d \equiv 0$, *mod.* 3, then either a and d or b and c satisfy this congruence and therefore also $\lambda \equiv 0$, *mod.* 3; if no one of $a, b, c, d \equiv 0$, then from (3) $ad \equiv -1$, $bc \equiv 1$, whence $b - c \equiv 0$, so that again $\lambda \equiv 0$. Thus in all cases $\lambda \equiv 0$, *mod.* 3.

The congruences *mod.* 16 are rather more troublesome. From (2) it follows at once that if $b = 2^k \times$ (odd integer), $c = 2^{k'} \times$ (odd integer), then $k' = k$, for $k = 1, 2, 3$, and $k' \geq 4$ for $k \geq 4$; hence for $k \geq 2$ (which is the same condition as $b \equiv 0$, *mod.* 8) $b - c \equiv 0$ *mod.* 16 and $\lambda \equiv 0$, *mod.* 8; if $k = 1$ or 2, $ad = bc + 1 \equiv 1$, *mod.* 4 and $a \equiv d \equiv \pm 1$, *mod.* 4, so that $a + d \equiv 2$, *mod.* 4. We now have $a(b - c) = (a + d)b - (ac + bd) \equiv (a + d)b$, *mod.* 16, by (2), whence $b - c = 2^{k+1} \times$ (odd integer), so that for $k = 1$, $b - c \equiv \pm 4$, *mod.* 16, $\lambda \equiv \pm 2$, *mod.* 8, and for $k = 2$, $b - c \equiv 8$, *mod.* 16, $\lambda \equiv 4$, *mod.* 8. As we have seen that $\lambda \equiv 0$, *mod.* 3 it follows that $\lambda = \pm 6, 12, 0$, *mod.* 24 and $\epsilon^\lambda = \pm i, -1, 1$ according as $k = 1$, $k = 2$, $k > 2$. The condition $k = 1$ (or $b \equiv 2$, *mod.* 4) can be replaced by the simpler condition $n \equiv -3$, *mod.* 8, for a^2 being the square of an odd number is necessarily of the form $8p + 1$, so that $b^2 = n - a^2 \equiv n - 1$ *mod.* 8, and then $b \equiv 2$, *mod.* 4 or $b \equiv 0$ *mod.* 4 according as $n \equiv -3, 1$, *mod.* 8.

The discrimination between the cases $k = 2$, $k > 2$ (or $b \equiv 4$, $b \equiv 0$, *mod.* 8), does not appear possible by means of any linear congruence for n but requires the actual expression of n ($n = 8p + 1$) in the form $a^2 + b^2$ or some equivalent process in the arithmetical theory of quadratic forms.

* Mathews, *Theory of Numbers*, § 91.

By the proof already given if one isolated root is known the other is its conjugate imaginary, so that if one is $\pm ix(i)$, $= \pm i2^{-1/4}$ the other is $\mp ix(i)$, and if one is $\pm x(i)$ the other is equal to it.

Thus we have the final result:

If $n \equiv -1$ *mod.* 4, all the roots correspond to branch points; if $n \equiv -3$, *mod.* 8, there are two isolated roots $\pm i2^{-1/4}$; if $n \equiv 1$, *mod.* 8, there are two isolated roots both $-2^{-1/4}$ or both $2^{1/4}$ according as, when n is expressed in the form $a^2 + b^2$ (a odd, b even), $b \equiv 4$ or $b \equiv 0$, *mod.* 8.

I take the opportunity of correcting some errata in my former paper:

P. 158, line 11, *for* mod. 48 *read* mod. 24.

P. 165, line 18, *for* $x\{(-1 + i\sqrt{27})/31\}$ *read* $x\{(-2 + i\sqrt{27})/31\}$.

P. 165, line 23, *for* 24 *read* 2^4.

P. 165, lines 23, 26, *for* $\sqrt{19}$ *read* $\sqrt{15}$.

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ASSOCIATED FORMS IN THE GENERAL THEORY OF MODULAR COVARIANTS.

By Olive C. Hazlett.

1. **Historical Background.**—In any theory of covariants, it is of prime importance to ascertain whether or not all covariants of the set are expressible as functions of the covariants belonging to a finite subset. We may attack this fundamental problem from either one of two different points of view: either we may endeavor to express all covariants of the set as rational integral functions of the covariants of a finite subset, or we may content ourselves with the problem of finding rational relations (syzygies) connecting the covariants. The first leads to the finiteness theorem; the second, to the theory of associated forms. Whichever problem we attack, there emerge two entirely different theories, according as the coefficients of the transformations of the group are marks of a field of characteristic zero or marks of a field of characteristic $p \neq 0$.

In the theory of algebraic covariants of a system of forms under a group of transformations having the coefficients in a field of characteristic zero, both problems have been successfully attacked. The most important names to be associated with the first problem are, Gordan, Mertens, and Hilbert;* with the second problem, we associate the names of Boole, Hermite and Clebsch.†

In the theory of algebraic covariants of a system of forms under a group of linear transformations whose coefficients are marks of a field of character $p \neq 0$, comparatively little has as yet been accomplished on either of these problems. It must be remembered that, in the case of a finite field, there present themselves two distinct kinds of algebraic covariants in contrast to the single kind of algebraic covariant that arises when the field is of characteristic zero. For, in the latter field, if a function be unaltered in form, it is unaltered in value and conversely. Whereas, in a finite field, if a function be unaltered in form it is unaltered in value, but the converse

* Gordan, numerous articles in the journals (for references, see the German encyclopedia, IB2, § 6) especially *Journal für Mathematik*, Vol. 69 (1868), pp. 323–354, *Mathematische Annalen*, Vol. 2 (1870), pp. 227–280. Also his "Invariantentheorie," Vol. 2, pp. 231–236; Mertens, *Journal für Mathematik*, Vol. 100 (1887), pp. 223–230; Hilbert, *Mathematische Annalen*, Vol. 36 (1890), pp.473.

† Boole, *Cambridge Mathematical Journal*, Vol. 3 (1841), pp. 1–20, 106–119; Hermite, *Journal für Mathematik*, Vol. 52 (1856), pp. 1–38; Clebsch, *Mathematische Annalen*, Vol. 3 (1871), pp. 265–267, and "Theorie der binären algebraischen Formen" (1872), p. 410.

is not true in general, in view of Galois' generalization of Fermat's theorem.* Accordingly, then, we distinguish two kinds of algebraic covariants of a system of forms under a group of linear transformations with coefficients in the Galois field $GF[p^n]$—formal (modular) covariants and modular covariants, according as the coefficients of the original form are regarded as independent variables or as marks of the field.

The first to consider formal invariants was Hurwitz,† who found that they arose naturally in an inquiry into the number of roots of the congruence $a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x_1 + a_0 \equiv 0 \pmod{p}$. He proved the finiteness theorem for formal invariants for the special case where the order of the group G is not divisible by p‡. This case is of only minor importance; for the total linear group of transformations whose coefficients are in the field $GF[p^n]$ is of order $p^n(p^n-1)(p^{2n}-1)$ which is congruent to zero modulo p. Four years later, Dickson introduced the notion of modular invariants§ and published an elegant theory of modular invariants in which he proved that there is only a finite number of modular invariants of any system of forms under any group G of linear transformations.|| Four years later, Dickson proved that the set of all modular covariants of any system of forms possesses the finiteness property, i.e., they are all expressible as polynomials in the covariants belonging to a finite subset.** In 1914, one of Dickson's students extended this theorem to the modular invariants of a system of forms and a number of cogredient binary points.†† Up to the present, the finiteness theorem has been proved for formal covariants only in very special cases—these are due to Professor Glenn.‡‡ As Hurwitz pointed out, this is a most difficult problem, for none of the methods that obtain in the classical theory of algebraic covariants will apply here.

* If a is a mark of a finite field F of order p^n, then $a^{p^n} \equiv a$ in the field. In case $n = 1$ the marks of the field are the classes of residues of integers reduced modulo p, and Galois' theorem reduces to Fermat's theorem.

† "Ueber höhere Congruenzen," *Archiv der Mathematik und Physik*, series 3, Vol. 5 (1903), pp. 17–27.

‡ Loc. cit., p. 25.

§ "Invariants of binary forms under modular transformations," *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 205–232.

|| "General Theory of Modular Invariants," *Transactions of the American Mathematical Society*, Vol. 10 (1909), pp. 123–158. This is the basic paper on modular invariants.

** "Proof of the Finiteness of Modular Covariants," *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 299–310.

†† F. B. Wiley, "Proof of the Finiteness of the Modular Covariants of a System of Binary Forms and Cogredient Points," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 431–438.

‡‡ "A Fundamental System of Formal Covariants Modulo 2 of the Binary Cubic," *Transactions of the American Mathematical Society*, Vol. 19 (1918), pp. 109–118; "Modular Concomitant Scales, with a Fundamental System of Formal Covariants, Modulo 3, of the Binary Quadratic," *Transactions*, Vol. 20 (1919), pp. 154–168.

Hence, since this problem is so intractable, it may be of interest to consider the related problem of syzygies. The present paper extends the method and results of Hermite's fundamental memoir on associated forms for ordinary algebraic covariants to modular covariants, both formal and otherwise. The main theorem proves that, if $f = a_0x_1 + a_1x^{m-1} + \cdots$ is a binary form of order not divisible by p, then any modular covariant of f for the Galois Field $GF[p^n]$ of order p^n is expressible (aside from a power of f) as a polynomial in the universal covariants Q and L, where the coefficients of the terms in Q are polynomials in the forms associated with f. We also prove an analogous theorem for formal covariants. From the first of these we prove the rather striking corollary that, aside from a power of f, every modular covariant is congruent to an ordinary algebraic covariant of f whenever the variables x and y are in the field. Similarly we prove that, aside from a power of a_0, every modular invariant is congruent to an ordinary algebraic invariant of f. Another corollary gives a neat method of constructing a modular covariant having a given leader, provided the leader has a_0 as a factor. The main theorem, together with its corollaries, is verified for the binary quadratic, modulo 3.

2. Hermite's Two Propositions.—In a fundamental memoir* already mentioned, Hermite proves the following

FIRST PROPOSITION. *Let g and h be any two algebraic covariants of the form $f(x, y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m$ of indices s and t respectively, and let us set*

$$g\left(a_0, a_1, a_2, \cdots;\ xX - \frac{\partial h}{\partial y}Y,\ yX + \frac{\partial h}{\partial x}Y\right) \tag{1}$$
$$= \theta(a_0, a_1, a_2, \cdots;\ x, y;\ X, Y).$$

Then, under the linear transformation

$$\left.\begin{aligned} x &= \alpha x' + \beta y' \\ y &= \gamma x' + \delta y' \end{aligned}\right\} \qquad \Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0, \tag{2}$$

we have the following identity

$$\theta(a_0', a_1', \cdots;\ x', y';\ X, Y) = \Delta^s\theta(a_0, a_1, \cdots;\ x, y;\ X, \Delta^{t+1}Y). \tag{3}$$

This means that the coefficients of the various terms in X and Y in the function θ are covariants of F. In case we take f as the function g, the covariants of f thus obtained he calls the *covariants associated with f.*

If d is the order of the covariant h, then we can write $(1/d)Y$ instead of Y in (1) and (3). If, furthermore, we take as our new variables $xX + x'Y$ and $yX + y'Y$ where $x' = (1/d)(\partial h/\partial y)$ and $y' = (1/d)(\partial h/\partial x)$, the deter-

* "Sur la théorie des fonctions homogènes à deux indéterminées," *Journal für die reine und angewandte Mathematik*, Vol. 52 (1856), pp. 21–23.

minant $xy' - yx'$ of this transformation is the form h itself. The coefficient of X^m will be f itself and the other coefficients will be denoted by h_1, h_2, $\cdots$, h_m; that is,

$$f\left(a_0, a_1, \cdots \quad xX - \frac{1}{d}\frac{\partial h}{\partial x}Y,\ yX + \frac{1}{d}\frac{\partial h}{\partial x}Y\right) = f(f,\ h_1,\ h_2,\ \cdots,\ h_m;\ X,\ Y).$$

Thus, if C is any ordinary algebraic covariant of index w, then

$$h^w C\left(a_0, a_1, \cdots;\ xX - \frac{1}{d}\frac{\partial h}{\partial y}Y,\ yX + \frac{1}{d}\frac{\partial h}{\partial x}Y\right) = C(A_0, A_1, \cdots;\ X, Y),$$

where the A_0, A_1, $\cdots$ are respectively f, h_1, $\cdots$, h_m. If we now set $X = 1$ and $Y = 0$, this gives us

$$h^w C(a_0, a_1, \cdots;\ x, y) = C(f, h_1, \cdots, h_m;\ 1, 0).$$

Thus he proves his

SECOND PROPOSITION. *Every ordinary algebraic covariant of f, when multiplied by a suitable integral power of h becomes a rational integral function of the associated covariants.*

3. **Extension to Modular Covariants.**—If we now assume that g and h are modular covariants (either formal or otherwise), and follow through Hermite's proof of the First Proposition, we readily see that in this case we obtain the following result,—if g and h are any two modular covariants of f and θ is defined by (1), then when x and y are subjected to the transformation (2) where α, β, γ and δ are marks of the Galois Field $GF[p^n]$ of order p^n, then

$$(4)\quad \theta\left(a_0', a_1', \cdots;\ x', y';\ X, \frac{1}{\Delta^{t+1}}Y\right) \equiv \Delta^s\theta(a_0, a_1, \cdots;\ x, y;\ X, Y).$$

That is,

$$\theta\left(a';\ x', y';\ X, \frac{1}{\Delta^{t+1}}Y\right) = \Delta^s\theta(a;\ x, y;\ X, Y) + \varphi,$$

where φ vanishes whenever α, β, γ and δ are in the field. Here it must be noted that X and Y are indeterminates. Thus we have

THEOREM I. *The coefficients of the different terms in X and Y are modular covariants (formal or otherwise). If g and h are both formal covariants, then these coefficients are formal covariants; if, however, either g or h is not formally covariant, then the coefficients are not all formal covariants.* In case g is the form f itself, then the covariants obtained in this manner we shall call the *associated covariants of f with respect to the covariant h.*

The above result still holds if we replace $\partial h/\partial x$ and $\partial h/\partial y$ by $\frac{1}{d}\partial h/d\partial x$ and $\frac{1}{d}\partial h/d\partial y$ respectively, where d is the order of h, provided d is not

divisible by p. Now for every form f there is always a function h which is covariant under the total group of linear transformations whose coefficients are marks of the field $GF[p]$ and whose order d is not divisible by p.* First, every binary form of odd order > 3 has a (non-vanishing) ordinary algebraic covariant of order one and every binary form of even order > 4 has a covariant of order 2.† Hence we can confine our attention to binary forms of order 3 where $p = 3$ and binary forms of orders 2 and 4 where $p = 2$. Now a binary cubic has an ordinary quadratic covariant.‡ Moreover, the quadratic and quartic modulo 2 have a modular covariant of order one.§ Hence every binary form has with respect to the field $GF[p]$ a modular covariant which is of order $d \neq 0 \pmod{p}$; this is also known to be true with respect to the field $GF[p^n]$ where $n > 1$, provided $p \neq 2$.

Now let C be any modular covariant of f of index w. Without loss of generality, we may assume that C is homogeneous in x and y and pseudo-homogeneous‖ in the $a's$, since any covaraint is the sum of a finite number of such covariants. If we replace x and y by $xX + x'Y$ and $yX + y'Y$ respectively, where $xy' - x'y \neq 0$, and x, y, x', y' are indeterminates, then

$$C(A's;\ X,\ Y) \equiv (xy' - x'y)^w C(a's;\ xX + x'Y,\ yX + y'Y)$$

whenever x, y, x', y' are in the field, provided $xy' - x'y \neq 0$. If we now take

$$(6) \qquad x' = -\frac{1}{d}\frac{\partial h}{\partial y} \qquad y' = \frac{1}{d}\frac{\partial h}{\partial x} \qquad (d \neq 0, \bmod p),$$

then the $A's$ become the associated covariants of f with respect to h, and the determinant of the transformation is $h(x, y)$. Hence, on substituting $X = 1$ and $Y = 0$, we find that $C(A's;\ 1,\ 0) \equiv [h(x,\ y)]^w C(a's;\ x,\ y)$ whenever x and y are in the field and $h(x, y) \neq 0$. In case the left member is divisible by $h(x, y)$ and $w > 1$, this congruence is true even when $h(x, y) \equiv 0$; in order to have a congruence which is true without restriction, we multiply throughout by $h(x, y)$ and thus have

$$(7) \qquad h(x,\ y)C(A's;\ 1,\ 0) \equiv [h(x,\ y)]^{w+1} C(a's;\ x,\ y)$$

whenever x and y are in the field.

* Loc. cit., pp. 23–24.

† Clebsch, "Theorie der binäre algebraischen Formen," p. 410; Elliott, "Algebra of Quantics," 1st edition, p. 74.

‡ Gordan, "Invariantentheorie," p. 167; Weber, "Lehrbuch der algebra," Vol. I, p. 223; Elliott, p. 109.

§ Dickson, Madison Colloquium Lectures, p. 56; Glenn, "A System of Formal Covariants, Modulo 2, of the Binary Cubic," *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 110.

‖ We say that a function $\varphi(x, y, \cdots)$ is pseudo-homogeneous of degree d in the Galois Field $GF[p^n]$ of order p^n if $\varphi(\lambda x, \lambda y, \cdots) \equiv \lambda^d \varphi(x, y, \cdots)$ whenever λ is in the field. In case φ is a polynomial, this simply amounts to saying that the degrees of any two terms of φ differ at most by an integral multiple of $p^n - 1$.

First let us consider the case in which C is a non-formal modular covariant. Then, in order to obtain from (7) a congruence which shall be an identity in x and y, we must make (7) homogeneous in x and y. Now the right side is already homogeneous. The order of the left side is determined by the weight of C and the degree of C in the a's. But since the weight of any two terms of any modular covariant (such as C) differ at most by integral multiples of $p^n - 1$, it follows that the orders of any two terms B_1 and B_2 of $C(A's;\ 1, 0)$ can differ at most by an integral multiple of $p^n - 1$. Let the degree of B_1 in x and y be ω_1 and let the degree of B_2 be ω_2, where $\omega_1 - \omega_2 = k(p^n - 1)$ and k is a positive integer. Then there is a formal covariant P such that $B_2' = B_2P^k$ is congruent to B_2 whenever x and y are in the field such that $h(x, y) \neq 0$, and B_2' is of the same degree as B_1 in x and y. We may take $P = Q^\alpha/H^\beta$ where Q is the universal covariant $(x^{p^{n2}}y - xy^{p^{n2}})/L$, $H = [h(x, y)]^{p^n-1}$, and α and β are two positive integers such that $\alpha p^n - \beta d = 1$. Thus, by multiplying each term of the left member of (7) by a suitably high power of P, we can make the left member homogeneous in x and y of a degree greater than that of the right side. Then, if we multiply the right side of (7) by a suitable power of $P' = H^{\beta'}/Q^{\alpha'}$ where α', β' is a pair of positive integers such that $\beta' d - \alpha' p^n = 1$, we can make the right side of (7) of the same degree as the left side.† Thus, by multiplying both sides of the congruence just obtained by a suitable power of P', we obtain from (7) a congruence which is equivalent to (7) whenever x and y are in the field. This congruence we may write

$$(8) \qquad [h(x, y)]^r C(a;\ x, y) \equiv \mathfrak{C}(Q, h, A's).$$

Since this is homogeneous in x and y and holds for all values of x and y in the field, we have the identity

$$(9) \qquad [h(x, y)]^r C(a;\ x, y) \equiv \mathfrak{C}(Q, h, A's) + LC_1$$

where C_1 is a modular covariant and $\mathfrak{C}$ is a polynomial in its arguments.

Hence we have proved

THEOREM II. *Let $f(x, y)$ be any modular form with coefficients $a_0, a_1, \cdots,$ in the Galois field $GF[p^n]$ of order p^r, and let $h(x, y)$ be any modular covariant of $f(x, y)$—whose order $\neq 0$—under any group G of linear transformations with coefficients in the field. If $C(a's;\ x, y)$ is any modular covariant of $f(x, y)$, then—aside from a power of h—$C(s's;\ x, y)$ is equal to a polynomial in Q and the associated covariants of f with respect to h plus L times a modular covariant. Observe that, unless $p = 2$, there is always an ordinary algebraic*

* Dickson, "Invariants of Binary Forms under Modular Transformations," *Transactions of the American Mathematical Society*, Vol. 8 (1907), p. 209.

† Note that we can take α and β such that $\alpha < d$ and $\beta < p^n$. With this choice of α and β, we can take $\alpha' = -\alpha + d$, $\beta' = -\beta + p^n$.

covariant satisfying the above conditions for h; and hence, unless $p = 2$, the associated covariants of f with respect to h are all ordinary algebraic covariants of f.

4. **Several Corollaries.**—If, in Theorem II, we take $h = f$ and $x = 1$, $y = 0$, we have

COROLLARY I. *Aside from a power of a_0, every modular seminvariant of a binary form f whose order is not divisible by p is a polynomial in the ordinary seminvariants of f.*

Observe that this is not equivalent to saying that, aside from a power of a_0, every modular seminvariant of f is an ordinary seminvariant of f; for, in order that a polynomial in ordinary seminvariants be an ordinary seminvariant, it must be homogeneous in the a's and isobaric, whereas, a modular seminvariant is not necessarily either homogeneous or isobaric.

We can also derive a method of constructing a modular covariant having a given seminvariant as leader. For, if $C(a's;\ x,\ y)$ is any function having the invariantive property under a transformation T, then equation (5) holds provided x and y have such values that

$$\bar{x} = xX - \frac{1}{d}\frac{\partial h}{\partial y}Y, \qquad \bar{y} = yX + \frac{1}{d}\frac{\partial h}{\partial x}Y,$$

is the transformation T. In case C is a seminvariant, (5) must be true whenever $y = 0$ and x is in the field. Thus for a seminvariant, (7) and (8) hold when $y = 0$ and x is in the field. Since equation (8) is homogeneous in x and y, this means that the leader of $\mathfrak{C}$ is C times the r'th power of the leader of h. In case the leader of h is a factor of the seminvariant C, then the leader of $h^{p^n-r-1}\mathfrak{C}$ is C. In case the order of the form f is not divisible by p, we may take $h = f$, and thus we prove the following

COROLLARY II. *Let $f(x,\ y)$ be a binary form of order $m \not\equiv 0 \pmod{p}$, and let $S(a_0,\ a_1,\ \cdots)$ be any modular seminvariant of f with respect to the Galois field $GF[p^n]$ of order p^n. If S contains a_0 as a factor, then a modular covariant having the given seminvariant as leader is obtained by making $S(A_0,\ A_1,\ \cdots)$ homogeneous in x and y by the method used in § 3 and then multiplying by a suitable power of a_0. Here $A_0,\ A_1,\ \cdots$ are the ordinary algebraic covariants which Hermite called the associated covariants of f.*

5. **Theorem for Formal Covariants.**—In case $C(a's;\ x,\ y)$ is a formal modular covariant of a binary form $f(x,\ y)$, we have

$$h(x,\ y)C(A's;\ 1,\ 0) \equiv [h(x,\ y)]^{w+1}C(a's;\ x,\ y) \tag{7}$$

holding whenever x, y and the a's are all in the field. Since C is a formal covariant, we may without loss of generality assume that C is homogeneous in x and y and homogeneous in the a's. Then the left member of (7)

is pseudo-homogeneous in x and y and pseudo-homogeneous in the a's. We now have the double task of making (7) homogeneous in the a's as well as homogeneous in x and y. To do this, first make (7) homogeneous in x and y by multiplying each term by a suitable power of P, just as in § 3. Thus we have

$$(9) \qquad [h(x, y)]^r C(a;\ x, y) \equiv \mathfrak{C}(Q, h, A's) + LC_1,$$

holding whenever the a's are marks of the field. Now any two terms of (9) are of the same order, but of different degrees in the a's, their degrees differing at most by an integral multiple of $p^n - 1$. Let the difference between the lowest degree and the highest degrees be $k(p^n - 1)$. Then multiply (9) throughout by $[h(x, y)]^k$. If $h'(x, y)$ denote the polynomial obtained from $h(x, y)$ by replacing $a_0, a_1, \cdots$ by $a_0^{p^n}, a_1^{p^n}, \cdots$, then $h'(x, y) \equiv h(x, y)$ when the a's are marks of the field. Moreover $h'(x, y)$ is a formal covariant.* Hence we may replace h in (9) wherever we wish by h' without changing the validity of the congruence or the formal covariancy. If, in any term we replace h^l by h'^l where the degree of that term is $l(p^n - 1)$ less than the maximum degree of any such term, we obtain a congruence which we may write

$$(10) \quad [h(x, y)]^q [h'(x, y)]^{r-q} C(a;\ x, y) \equiv \mathfrak{C}'(Q, h, h', A's) + LC_1' + K$$

where K is a formal covariant (since all the other functions in the congruence are formal covariants) which is congruent to zero whenever the a's are in the field. Thus once more we see the importance of those irreducible formal covariants which are congruent to zero for all values of the coefficients in the field.† Thus we have proved

THEOREM III. *Let $f(x, y)$ be any form with coefficients $a_0, a_1, \cdots$ which are indeterminates, and let $h(x, y)$ be any formal covariant of $f(x, y)$ under a group of linear transformations having coefficients in the Galois field $GF[p^n]$ of order p^n. Moreover, let the order of $h(x, y)$ be not divisible by p. If $C(a's;\ x, y)$ is any formal covariant of $f(x, y)$, then if we multiply $C(a's;\ x, y)$ by a suitable power of $h(a's;\ x, y)$ and by a suitable power of $h(a^{p^n};\ x, y)$ the result is identically congruent to a polynomial in Q, L and the irreducible formal covariants which vanish whenever the a's are in the field. The coefficients of the different powers of Q are polynomials in $h(a's;\ x, y)$, $h(a^{p^n};\ x, y)$ and the formal covariants which are the associated covariants of $f(x, y)$ with respect to $h(x, y)$.* Notice that, unless $p = 2$, the covariant $h(x, y)$ can be taken as an ordinary algebraic covariant.

* O. E. Glenn, "Modular Invariant Processes," *Bulletin of the American Mathematical Society*, Vol. 21 (1915), pp. 167–173.

† In a recent paper (*Transactions of the American Mathematical Society*, Vol. 22 (1921), April issue) the writer proved that the set of all formal covariants of a system of forms S has the finiteness property if and only if the finiteness property is possessed by the set of those formal covariants of S which vanish whenever the coefficients of S are marks of the field.

6. Verification for the Binary Quadratic, Modulo 3.—Dickson* has shown that a fundamental set of modular covariants of the binary quadratic, $f_2 = a_0x_1^2 + 2a_1xy + a_2y^2$ (mod 3) consists of

$$f_2, \qquad f_4 = a_0x^4 + a_1x^3y + a_1xy^3 + a_2y^4,$$

$$L = x^3y - xy^3, \qquad Q = x^6 + x^4y^2 + x^2y^4 + y^6,$$

$$\Delta = a_1^2 - a_0a_2, \qquad q = (a_0 + a_2)(a_1^2 + a_0a_2 - 1),$$

$$C_1 = (a_0^2a_1 - a_1^3)x^2 + 2(a_1^2 + a_0a_2)(a_2 - a_0)xy + (a_1^3 - a_1a_2^2)y^2,$$

$$C_2 = (\Delta + a_0^2)x^2 - 2a_1(a_0 + a_2)xy + (\Delta + a_2^2)y^2.$$

By Hermite's fundamental memoir, the associated covariants of f_2 are $A_0 = f_2$, $A_1 = 0$, $A_2 = -\Delta f_2$. Theorem II is clearly true for polynomials in f_2 and Δ, since these are ordinary algebraic covariants.

If we take $C = f_4$, then since here $C(A's;\ 1, 0) = A_0 = f_2$, (7) becomes $f_2 \equiv f_4$ (whenever x and y are in the field) and (8) becomes $f_2^3f_4 \equiv f_2^2Q - L^2C_2$. When $C = C_1$, we find that (8) becomes† $0 \equiv f_2C_1 + (\Delta^2 + \Delta)L$. For $C = C_2$, (8) becomes $(1 + \Delta)f_2^2Q \equiv f_2^4C_2 + L^2(qf_2 - C_2)$. Thus the theorem is verified when C is any polynomial in Δ, f_2, f_4, C_1, C_2 and L, Q. Is it true for any covariant containing q as a factor, say qQf_2? For $C = qQf_2$, $C(A's;\ 1,\ 0) = (f_2 + \Delta f_2)(\Delta f_2^2 - 1)f_2Q \equiv (\Delta^2 - 1)f_2^4$ when x and y are in the field, and thus we have $qQf_2 \equiv (\Delta^2 - 1)f_2^4 - q^2L^2$.

7. Relation to the Literature.—The chief interest of the results of this paper lies in the relation shown to exist between modular covariants and ordinary algebraic covariants of the form. From a different point of view, the results are of interest in connection with several papers by Professor Glenn.

He considered the expansion of a homogeneous binary form in terms of two binary forms of lower order.‡ He found that a binary form f of order $n(m + 1) - 1$ can be expressed in the form

$$f = \varphi_{0p}f_{1n}^m + \varphi_{1p}f_{1n}^{m-1}f_{2n} + \cdots + \varphi_{mp}f_{2n}^m$$

(where f_{1n} and f_{2n} are two binary forms of orders n and the φ's are binary forms of order p) provided the resultant R of f_{1n} and f_{2n} does not vanish. Moreover this expansion is unique for any such pair of forms f_{1n}, f_{2n}.

* *Transactions of the American Mathematical Society*, Vol. 14 (1913), p. 310.

† This syzygy was given by Dickson, "Finiteness of Modular Covariants," *Transactions of the American Mathematical Society*, Vol. 14 (1913), p. 310.

‡ "The Symbolical Theory of Finite Expansions," *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 72–86.

He also showed that a form f of order m can be expressed linearly in terms of two forms f_{1n_1} and f_{2n_2} of different orders n_1 and n_2 respectively, provided the resultant of f_{1n_1} and f_{2n_2} is different from zero.* Thus

$$f_m = \varphi_{1m-n_1} f_{1n_1} + \varphi_{2m-n_2} f_{2n_2} \tag{11}$$

where φ_{1m-n_1} and φ_{2m-n_2} are of orders $m - n_1$ and $m - n_2$ respectively. If $n_1 + n_2 = m + 1$, the expansion is unique.

If f_{1n_1} and f_{2n_2} are covariants of f and the expansion is unique, then the forms φ_{1m-n_1} and φ_{2m-n_2} are covariants of f. If, furthermore, (11) is a congruence modulo p a prime, and $f_{1n_1} = Q$, $f_{2n_2} = L$ where

$$L = x^p y - xy^p, \qquad Q = (x^{p^2} y - xy^{p^2}) \div L = x^{p(p-1)} + \cdots + y^{p(p-1)},$$

then we have

$$f_m \equiv Q\varphi_1 + L\varphi_2 \quad (\text{mod } p) \tag{12}$$

and it appears that any binary form of order $m = p^2$ is reducible in terms of Q and L and two first degree formal modular covariants with respect to the $GF[p]$ whose orders are respectively p and $p^2 - p - 1$.† As Professor Glenn points out, "this raises the question as to whether the modular expansions (12) for $m > p^2$, containing arbitrary parameters in their coefficients may have these parameters determined so that the coefficient forms φ_{1m-p^2+p}, φ_{2m-p-1} are modular covariants." For various cases in which $m > p^2$, he has determined modular covariants φ_1 and φ_2 of f such that the congruence (12) will hold identically in the a's and the variables x, y.

Theorem II of the present paper shows that, for any Galois Field $GF[p^n]$ of order p^n, if a modular covariant C of f be multiplied by a sufficiently high power of h, then it is expressible in the form $Q\varphi_1 + L\varphi_2$ where φ_1 and φ_2 are modular covariants of f. Moreover, φ_1 is a polynomial in Q, f, h and the associated covariants of f with respect to h. At present I do not know whether it is true that, for every covariant of f, there is one such expansion in which φ_2 is of lower order than C. If this be so, then it would follow by induction that, aside from a power of h, every modular covariant of f is a polynomial in Q, L, f, h and the associated covariants of f with respect to h.

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* "A Memoir on the Doctrine of Associated Forms," *Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 443–462 (especially pp. 446–448).

† See article referred to in previous note, p. 461.

ON (2, 3) COMPOUND INVOLUTIONS.

By Temple Rice Hollcroft.

1. *Introduction.*—This paper together with a preceding one* is intended to complete the discussion and classification of (2, 3) point correspondences between two planes.

In the general point correspondence a point and its successive images do not form a closed group. In a compound involution, however, the correspondence closes up at the second application of the transformation. Choosing (x) and (x') respectively as the double and triple planes, to a point P' of (x') correspond three points P_1, P_2, P_3 of (x) such that to each of these image points correspond the original point P' and a residual point P'_0 of (x'). To P'_0 as well as to P' correspond the three points P_1, P_2, P_3 of (x). The point correspondence thus established is involutorial. The essential difference between this correspondence and the general (2, 3) point correspondence is that now the residual images coincide.

The first known paper discussing point correspondences with both planes multiple was published in 1889.† This is a very simple (2, 2) compound involution in which the lines of either plane correspond to conics of the other. Burali-Forti‡ later obtained certain (2, 2) compound involutions by combining two (1, 2) involutions and showed that the case treated by Visalli is included in these. Finally the (2, 2) compound involutions have been classified and six independent types obtained by F. R. Sharpe and Virgil Snyder.§

2. *Outline of Method.*—Suppose that a (1, 2) point correspondence has been established between the planes (x') and (y'). Then to a point P' of (y') correspond two points P'_1, P'_2 of (x') and to either P'_1 or P'_2 of (x') corresponds the one point P' of (y'). Also to the lines through P'_1 or P'_2 correspond rational curves of (y') all passing through P'.

Assume further that a (1, 3) point correspondence exists between the planes (x) and (y). Then to a point P of (y) correspond three points

* T. R. Hollcroft: "A Classification of General (2, 3) Point Correspondences between two Planes," Am. Jour. of Math., Vol. XLI (1919), pp. 5–24.

† P. Visalli, "La trasformazione quadratica (2, 2)," *Rend. del Circ. Mat. di Palermo*, Vol. 3 (1889), pp. 165–170.

‡ "Sulle trasformazione (2, 2) che si possono ottenere mediante due trasformazioni doppie," *Rend. del Circ. Mat. di Palermo*, Vol. 5 (1891), pp. 91–99.

§ "Types of (2, 2) Point Correspondences between two Planes," *Trans. Amer. Math. Soc.*, Vol. XVIII (1917), pp. 409–414.

P_1, P_2, P_3 of (x), to each of the image points P_1, P_2, P_3 corresponds P of (y) and to the lines through P_1, P_2 or P_3 correspond rational curves of (y) all passing through P.

Now consider the planes (y) and (y'). Both contain nets of rational curves through the points P and P' respectively. The planes are therefore birationally equivalent, that is, a (1, 1) correspondence exists between them such that P corresponds to P' and P' to P and such that curves in (y) which are images of the lines of (x) and curves in (y') which are images of the lines of (x') are transformed reciprocally one into the other.

Then to two points P'_1, P'_2 of $(x)'$ corresponds P' of (y') to which corresponds P of (y) to which correspond P_1, P_2, P_3 of (x). Reciprocally, P_1, P_2, P_3 of (x) go into P of (y), thence into P' of (y'), thence into P'_1, P'_2 of (x'). The planes (x) and (x') are therefore so related that to two points of (x') correspond three points of (x) and to these three image points correspond the original two of (x'), that is, a (2, 3) compound involution has been established between them.

There are other methods of establishing (2, 3) compound involutions, but it will be proved later that all the independent types of (2, 3) compound involutions can be obtained by the method outlined above.

3. *General Properties.*—The curves of either plane are transformed into curves of the other by three transformations, two rational and one irrational. Since the two planes (y) and (y') are birationally equivalent, they may be considered the same plane in which the basis points of the two systems are rationally separable. Then the image in (x') of a curve of (x) is obtained by applying to its image in (y), considered as being also in (y') and retaining its basis points as fixed points of (y'), the transformation from (y') to (x'). Thus if a line of (x) goes into $C_n(y)$ with basis points kP_i* and a line of (y') corresponds to $C'_m(x')$ with basis points lP'_j, then the line of (x) corresponds to $C'_{mn}(x')$ with basis points lP'_{jn} and $2kP'_i$. A similar series of transformations relates (x') to (x).

If the two images P'_1, P'_2 of a point P of (x) coincide, P is on $L(x)$ the branch-point curve of (x). The locus of the corresponding coincidences is $K'(x')$, the coincidence curve of (x'). $K'(x')$ is the jacobian of the net of image curves of the lines of (y'). Its complete image is $L(x)$. The image of $L(x)$ is $K'(x')$ counted six times.

The coincidence curve $K(x)$ is the jacobian of the net of image curves in (x). The residual curve $\Gamma(x)$ is the co-jacobian of this net. To a point P' on $L'(x')$ corresponds two coincident points on $K(x)$ and one on $\Gamma(x)$. The complete image of either $K(x)$ or $\Gamma(x)$ is $L'(x')$. The image of $L'(x')$ is $K(x)$ counted four times and $\Gamma(x)$ counted twice. The branch-point

* Hollcroft, loc. cit., page 7.

curves and their corresponding coincidence or residual curves are not in (1, 1) correspondence as in the case of (1, n) or general (2, 3) point correspondences.

The non-basic intersections of $K(x)$ and $\Gamma(x)$ are all contacts to each of which corresponds two cusps on $L'(x')$. The images of both the cusps of $L'(x')$ coincide at the point of tangency of $K(x)$ and $\Gamma(x)$. There are thus only a finite number of points of (x') whose three images in (x) coincide, namely, the cusps of $L'(x')$.

To a point of $K'(x')$ correspond three points of $L(x)$ to each of which corresponds the original point on $K'(x')$ counted twice. To a point of $K(x)$ correspond two points of $L'(x')$ to each of which correspond the original point of $K(x)$ counted twice and a point of $\Gamma(x)$, whose images are those same two points of $L'(x')$.

The non-basic intersections of $K'(x')$ with $L'(x')$, of $L(x)$ with $K(x)$ and of $L(x)$ with $\Gamma(x)$ are equal in number and are all contacts. To each tangency of $K'(x')$ with $L'(x')$ corresponds a tangency of $L(x)$ with $K(x)$ and of $L(x)$ with $\Gamma(x)$. To each of these two points corresponds the same point of contact of $L'(x')$ and $K'(x')$. Since $K'(x')$ is counted six times as the image of $L(x)$, each contact of $L'(x')$ with $K'(x')$ counts as six contacts. To four of these correspond the four coincident contacts of $(K)^4$ and L and to two of them, the two coincident contacts of $(\Gamma)^2$ and L.

4. *Types of* (1, 2) *Involutions.*—There are three independent types of (1, 2) point correspondences.* Type 1, the Geiser type, is obtained by the intersections of any line with an associated conic of a net, or by the cubics of a net through seven basis points. Type 2, the Jonquières type, is given by the intersections of a line of the pencil P with a curve of order n of a net having an $(n - 2)$-fold point at P. Type 3, the Bertini type, is given by the variable intersections of a cubic of a pencil with an associated sextic having double points at eight of the basis points of the pencil.

5. *Types of* (1, 3) *Involutions.*—Five independent types of (1, 3) point correspondences have been recently found.† Type 1 is given by the intersections of a plane field of lines with the associated cubics of a net; Type 2 by the intersections of a line pencil and a net of curves of order n with $(n - 3)$-fold points at the vertex of the pencil; Type 3 by the intersections of two conics of a system of conics through one basis point; Type 4 by the intersections of two cubics of a net of cubics through six basis points; Type 5 by the intersections of a cubic of a pencil with an associated curve of a system of curves of order nine having triple points at eight of the nine basis points of the pencil of cubics.

* See Pascal's (German) "Repertorium der höheren Mathematik," second edition, Vol. 2, pp. 366–370.

† Anna Mayme Howe, "A Classification of Plane Involutions of Order Three," AM. JOUR. OF MATH., Vol. XLI (1919), pp. 25–40.

6. *Types of* (2, 3) *Compound Involutions.*—Fifteen types of (2, 3) compound involutions are obtained by combining the types of (1, 2) and (1, 3) involutions. For convenience these have been divided into three classes according to the (1, 2) involution employed. There are five types in each class, namely, the five (1, 3) involutions combined with the (1, 2) involution determining that class. The following table shows the combination and type number for each type. The symbol C_n; jP_i denotes a curve of order n with j basis points each of multiplicity i. The arabic numerals are the type numbers. The type is established by combining the systems in the row and column in which the type number is found.

Class	Double Plane / Triple Plane	C_1; C_3	C_1, P_1; C_m, P_{m-3}	C_2, P_1; C_2, P_1	$C_3, 6P_1$; $C_3, 6P_1$	$C_3, 8P_1$; $C_9, 8P_3$
I	C_1; C_2	1	2	3	4	5
II	C_1, P_1; C_n, P_{n-2}	6	7	8	9	10
III	$C_3, 8P_1$; $C_6, 8P_2$	11	12	13	14	15

CLASS I.

7. *Type* 1.—In the (1, 2) point correspondence, denote the double plane by (y') and the simple plane by (x'). The defining equations of the Geiser type used in Class I may then be written,

(1) $\sum_{i=1}^{3} x'_i y'_i = 0,$

(2) $\sum_{i=1}^{3} y'_i v'_i(x') = 0,$

where $v'_i(x')$ are general conics of (x').

(The following notation will be used in describing the curve systems of the planes being discussed:

The symbol " $\sim$ " meaning " corresponds to ";

L, L', K, K', Γ, fixed curves as heretofore defined;

P, Q, P', Q', basis points;

$\bar{P}, \bar{P}'$ variable points;

p, genus of curve being described;

Subscripts of curves denote their order;

Subscripts of points denote their multiplicity.)

$C'_1(y') \sim C'_3(x')$; $p = 1$, $7P'_1$.

$C'_1(x') \sim C'_3(y')$; $p = 0$, $\bar{P}'_2$.

$K'_6(x')$; $p = 3$, $7P'_2$. $\quad L'_4(y')$; $p = 3$.

In the (1, 3) point correspondence, denote the triple plane by (y) and the simple plane by (x). The defining equations of Type 1 are:

(1) $\sum_{i=1}^{3} x_i y_i = 0,$

(2) $\sum_{i=1}^{3} y_i u_i(x) = 0$,

where $u_i(x)$ are general cubics of (x).

$C_1(y) \sim C_4(x)$; $p = 3$, $13P_1$.

$C_1(x) \sim C_4(y)$; $p = 0$, $\bar{P}_3$.

$K_9(x)$; $p = 15$, $13P_2$.

$\Gamma_{22}(x)$; $p = 15$, $13P_6$.

$L_{10}(y)$; $p = 15$, 21 cusps.

8. *Image Curves.*—In the (2, 3) compound involution established between the planes (x) and (x') by the two preceding involutions, to obtain the image in (x') of $C_1(x)$ we use the transformation from (y') to (x') on $C_4(y)$. $C_1(x) \sim C'_{12}(x') \equiv C_4(x'_2v'_3 - x'_3v'_2, x'_3v'_1 - x'_1v'_3, x'_1v'_2 - x'_2v'_1) = 0$. $C'_{12}(x')$ has $7P'_4$ given by $v'_1/x'_1 = v'_2/x'_2 = v'_3/x'_3$ and two variable triple points which are images of $\bar{P}_3$ of $C_4(y)$. C'_{12} is of genus 7.

The image of $C'_1(x)'$ is found by using the transformation from (y) to (x) on $C'_3(y')$. $C'_1(x') \sim C_{12}(x) \equiv C'_3(x_2u_3 - x_3u_2, x_3u_1 - x_1u_3, x_1u_2 - x_2u_1) = 0$. $C_{12}(x)$ has $13P_3$ given by $u_1/x_1 = u_2/x_2 = u_3/x_3$ and three variable double points, images of $\bar{P}'_2$ of $C'_3(y')$. C_{12} is of genus 13.

$L(x)$ is obtained by applying the transformation from (y) to (x) on $L'_4(y')$. This gives $L_{16}(x)$ of genus 27 with $13P_4$. The image of $L_{16}(x)$ in (y) is a curve of order 12 which is birationally equivalent to $L'_4(y')$ counted three times. The image of $L_{16}(x)$ is, therefore, $K'_6(x')$ counted six times. $K'_6(x')$ has $7P'_2$ and is of genus 3. The complete image of $K'_6(x')$ is $L_{16}(x)$.

$L'(x')$ is obtained by applying the transformation from (y') to (x') on $L_{10}(y)$. This gives $L'_{30}(x')$ with $7P'_{10}$, 42 cusps and of genus 49. The image of $L'_{30}(x')$ in (y') is a curve of order 20 which is birationally equivalent to $L_{10}(y)$ counted twice. Then the image of $L'_{30}(x')$ in (x) is $K_9(x)$ counted four times and $\Gamma_{22}(x)$ counted twice. $K_9(x)$ has $13P_2$ and is of genus 15. $\Gamma_{22}(x)$ has $13P_6$ and is of genus 15. The complete image of either $K_9(x)$ or $\Gamma_{22}(x)$ is $L'_{30}(x')$.

$K_9(x)$ and $\Gamma_{22}(x)$ have 21 tangencies to each of which correspond two cusps on $L'_{30}(x')$. Each of these two cusps corresponds to that same tangency counted as two points on K_9 and one on Γ_{22}.

Each pair, $K_9(x)$ and $L_{16}(x)$, $\Gamma_{22}(x)$ and $L_{16}(x)$, $K'_6(x')$ and $L'_{30}(x')$ has 20 contacts. The points of tangency of $K'_6(x')$ and $L'_{30}(x')$ correspond to those of $K_9(x)$ with $L_{16}(x)$ and of $\Gamma_{22}(x)$ with $L_{16}(x)$ and reciprocally.

9. *Successive Images of Lines.*—The image of $C'_1(x')$ as found is $C_{12}(x)$ with $13P_3$ and $3\bar{P}_2$. To $C_{12}(x)$ corresponds $C_9(y)$, birationally equivalent to $[C'_3(y')]^3$. The image of $C'_3(y')$ is $C_1(x)$ and a residual $C'_8(x')$ with $7P'_3$. Then to $C_{12}(x)$ corresponds $(C'_1)^3(C'_8)^3$ in (x'). The complete image of $C'_8(x')$ is $C_{12}(x)$. C'_8 and K'_6 have 6 nonbasic intersections at the intersections

of C_1' and K_6'. C_8' and L_{30}' have 30 non-basic intersections distinct from the intersections of C_1' and L_{30}'. C_{12} and L_{16} in (x) have 18 points of contact. These are the images of the 6 intersections of C_1' or C_8' with K_6'. Reciprocally, the images of these 18 tangencies are the 36 intersections of C_1' or C_8' with $(K_6')^6$, three tangencies corresponding to each set of 6 coincident intersections. To the 30 intersections of C_1' with L_{30}' and to the 30 intersections of C_8' with L_{30}' correspond the 30 intersections of C_{12} with K_9, also of C_{12} with Γ_{22} and reciprocally.

The image of $C_1(x)$ is $C_{12}'(x')$ with $7P_4'$ and $2\overline{P}_3'$. To C_{12}' corresponds $C_8'(y')$ which is birationally equivalent to $[C_4(y)]^2$. The image of $C_4(y)$ is $C_1(x)$ and a residual C_{15} with $13P_4$. Then $C_{12}'(x')$ corresponds to $(C_1)^2(C_{15})^2$ in (x). The complete image of $C_{15}(x)$ is $(C_{12}')^2$. C_{15} and L_{16} have 32 intersections which together with the 16 intersections of C_1 and L_{16} correspond to the 16 intersections of C_{12}' with K_6' and reciprocally. The curves C_{15} and K_9 have 31 and C_{15} and Γ_{22} have 18 intersections. C_{12}' and L_{30}' have 80 intersections to which correspond the 40 intersections of C_1 and C_{15} with K_9 and the 40 intersections of C_1 and C_{15} with Γ_{22}. The 80 intersections of C_{12}' and L_{30}' correspond to the 160 intersections of C_1 and C_{15} with $(K_9)^4$ and the 80 intersections of C_1 and C_{15} with $(\Gamma_{22})^2$.

The images of two lines $C_1'(x')$, $\overline{C}_1'(x')$ are $C_{12}(x)$, $\overline{C}_{12}(x)$ respectively, which have 27 non-basic intersections. The images of C_{12} and $\overline{C}_{12}$ are $(C_1')^3(C_8')^3$ and $(\overline{C}_1')^3(\overline{C}_8')^3$ respectively. C_8' and $\overline{C}_8'$ intersect in one non-basic point to which correspond three intersections of C_{12} and $\overline{C}_{12}$. To each of these three intersections correspond the intersection of C_8' with $\overline{C}_8'$ and the intersection of C_1' with $\overline{C}_1'$. These three intersections of C_{12} and $\overline{C}_{12}$ are the images of the intersection of the two lines of (x'). For the remaining 24 intersections of C_{12} and $\overline{C}_{12}$, to each of the three points in each set of eight corresponds a point of intersection of C_1' with $\overline{C}_8'$ and also an intersection of $\overline{C}_1'$ with C_8'. To these latter correspond respectively the 24 intersections of C_{12} with $\overline{C}_{12}$.

The preceding paragraph, with the exception of the order of the image curves of (x), applies equally well to all five types of Class I.

The images of two lines $C_1(x)$, $\overline{C}_1(x)$ are $C_{12}'(x')$, $\overline{C}_{12}'(x')$ respectively, intersecting in 32 points. The images of C_{12}' and $\overline{C}_{12}'$ are $(C_1)^2(C_{15})^2$ and $(\overline{C}_1)^2(\overline{C}_{15})^2$ respectively. C_{15} and $\overline{C}_{15}$ intersect in 17 non-basic points. To the intersection of C_1, $\overline{C}_1$ and two of the intersections of C_{15}, $\overline{C}_{15}$ correspond two intersections of C_{12}', $\overline{C}_{12}'$ and reciprocally. To the 15 intersections of C_1, $\overline{C}_{15}$, the 15 intersections of $\overline{C}_1$, C_{15} and the remaining 15 intersections of C_{15}, $\overline{C}_{15}$ correspond the remaining 30 intersections of C_{12}', $\overline{C}_{12}'$ and reciprocally. This paragraph, with the exception of the order of the image curves of (x'), applies equally well to types 6 and 11.

Since for all the types the details and methods are similar, only the results will be given for the remaining ones.

10. *Type* 2.—Type 2 of the (1, 3) point correspondences has the defining equations,

(1) $x_1y_1 + x_2y_2 = 0$,

(2) $y_1\varphi_1(x) + y_2\varphi_2(x) + y_3\varphi_3(x) = 0$,

where $\varphi_i(x)$ is a curve of order m with an $(m-3)$-fold point at the vertex of the line pencil, $Q \equiv (0, 0, 1)$. The basis point of (y) is $Q \equiv (0, 0, 1)$.

$C_1(y) \sim C_{m+1}(x)$, $p = 2m - 3$, Q_{m-2}, $(6m-6)P_1$.

$C_1(x) \sim C_{m+1}(y)$, $p = 0$, Q_m.

$K_{2m}(x)$; $p = 6m - 9$; Q_{2m-4}, $(6m-6)P_1$.

$\Gamma_{4m-2}(x)$; $p = 6m - 9$; Q_{4m-6}, $(6m-6)P_2$.

$L_{4m-2}(y)$; $p = 6m - 9$; Q_{4m-6}, $(6m-6)$ cusps.

For the (2, 3) compound involution:

$C'_1(x') \sim C_{3m+3}(x)$; $p = 6m - 5$; Q_{3m-6}, $(6m-6)P_2$, $3\overline{P}_2$.

$C_1(x) \sim C'_{3m+3}(x')$; $p = 2m + 1$; $7P'_{m+1}$; $2P'_m$.

$L_{4m+4}(x)$; $p = 8m + 3$; Q_{4m-8}, $(6m-6)P_4$.

$K_{2m}(x)$; $p = 6m - 9$; Q_{2m-4}, $(6m-6)P_1$.

$\Gamma_{4m-2}(x)$; $p = 6m - 9$; Q_{4m-6}, $(6m-6)P_2$.

$L'_{12m-6}(x')$; $p = 20m - 23$; $7P'_{4m-2}$, $2P'_{4m-6}$, $(12m-12)$ cusps.

$K'_6(x')$; $p = 3$; $7P'_2$.

11. *Type* 3.—Type 3 of the (1, 3) involutions has the defining equations:

(1) $\sum_{i=1}^{3} y_iu_i(x) = 0$,

(2) $\sum_{i=1}^{3} y_iv_i(x) = 0$.

wherein $u_i(x)$ and $v_i(x)$ are conics through the basis point Q.

$C_1(y) \sim C_4(x)$; $p = 2$; Q_2, $9P_1$.

$C_1(x) \sim C_4(y)$; $p = 0$; $\overline{P}_3$.

$K_9(x)$; $p = 9$; Q_5, $9P_2$.

$\Gamma_{14}(x)$; $p = 9$; Q_6, $9P_4$.

$L_8(y)$; $p = 9$; 12 cusps.

For the (2, 3) compound involution:

$C'_1(x') \sim C_{12}(x)$; $p = 10$; Q_6, $9P_3$, $3\overline{P}_2$.

$C_1(x) \sim C'_{12}(x')$; $p = 7$; $7P'_4$, $2\overline{P}_3$.

$L_{16}(x)$; $p = 23$; Q_8, $9P_4$.

$K_9(x)$; $p = 9$; Q_5, $9P_2$.

$\Gamma_{14}(x)$; $p = 9$; Q_6, $9P_4$.

$L'_{24}(x')$; $p = 33$; $7P'_8$, 24 cusps.

$K'_6(x')$; $p = 3$; $7P'_2$.

12. *Type* 4.—Type 4 of the (1, 3) point correspondences has defining

(x) is in a simple involution with the other two. Then we can map the plane (x) on a triple plane (y) by a $(1, 3)$ transformation and the plane (x') on a double plane (y') by a $(1, 2)$ transformation such that the two planes (y) and (y') are in $(1, 1)$ correspondence. For a $(1, 3)$ point correspondence can always be mapped on a triple plane by equations of the form,

(1) $\varphi_1(x)/y_1 = \varphi_2(x)/y_2 = \varphi_3(x)/y_3$,

where $\varphi_i(x) = 0$ define a net of curves with three variable intersections. The plane (y) is mapped on the plane (y') by the Cremona transformation,

(2) $ky'_i = f_i(y),\ i = 1, 2, 3.$

and thence on (x') by the transformation,

(3) $\psi'_1(x')/y'_1 = \psi'_2(x')/y'_2 = \psi'_3(x')/y'_3$,

wherein $\psi'_i(x') = 0$ defines a net of curves in (x') with two variable intersections. By means of (2) eliminate y_i and y'_i from (1) and (3) and we have the relation,

(4) $F_1(x)/F'_1(x') = F_2(x)/F'_2(x') = F_3(x)/F'_3(x')$

wherein any two F_i have three variable intersections and any two F'_i have two variable intersections. Since (4) is the necessary and sufficient condition that the curve system in either plane be a net, we have deduced the theorem:

The necessary and sufficient condition that a (2, 3) *point correspondence be a compound involution is that the image curves in either plane form a net.*

It may be here noted that when a pair of defining equations for a (2, 3) point correspondence have equations of the Bertini type locating the two images in the triple plane, the curve system of that plane forms a net and the point correspondence is always a compound involution.

27. Proof will now be given for the theorem:

The sufficient condition that a (2, 3) *point correspondence be a compound involution is that in either plane both components of the curve system defining the image points form pencils.*

Let the correspondence be defined by

(1) $u_1(x)u'_1(x') + u_2(x)u'_2(x') = 0,$

(2) $v_1(x)v'_1(x') + v_2(x)v'_2(x') = 0,$

such that the components of the curve system of (x) $[(x')]$ intersect in three [two] points. If both components of either system form a pencil both components of the other system also form a pencil because only two homogeneous parameters remain in each equation.

Choose any point (x'_1) of (x') and consider it as fixed. Then (x'_1) determines two curves of the system in (x') which intersect in another fixed point (x'_2). To (x'_1) correspond three fixed points (x_1), (x_2), (x_3) of (x) which lie at the intersection of two of the curves of the system in (x). Any one of the image points of (x) uniquely determines the other two, because their defining curves form pencils.

Since (x_1), (x_2), (x_3) are images of the fixed point (x'_1), the following relations hold:

$$
\begin{aligned}
(1)\quad & u_1(x_1)/u_2(x_1) = u'_1(x'_1)/u'_2(x'_1), \\
& v_1(x_1)/v_2(x_1) = v'_1(x'_1)/v'_2(x'_1), \\
(2)\quad & u_1(x_2)/u_2(x_2) = u'_1(x'_1)/u'_2(x'_1), \\
& v_1(x_2)/v_2(x_2) = v'_1(x'_1)/v'(x'_1), \\
(3)\quad & u_1(x_3)/u_2(x_3) = u'_1(x'_1)/u'_2(x'_1), \\
& v_1(x_3)/v_2(x_3) = v'_1(x'_1)/v'_2(x'_1).
\end{aligned}
$$

These relations give the relation in (x):

$$
\begin{aligned}
(\mathrm{I})\quad & u_1(x_1)/u_2(x_1) = u_1(x_2)/u_2(x_2) = u_1(x_3)/u_2(x_3), \\
& v_1(x_1)/v_2(x_1) = v_1(x_2)/v_2(x_2) = v_1(x_3)/v_2(x_3).
\end{aligned}
$$

The relation (I) means that any one of the three image points of (x) determines the other two—a fact already known. Likewise in (x') since the components form pencils and either image point determines the other, this relation must hold:

$$
\begin{aligned}
(\mathrm{I}')\quad & u'_1(x'_1)/u'_2(x'_1) = u'_1(x'_2)/u'_2(x'_2), \\
& v'_1(x'_1)/v'_2(x'_1) = v'_1(x'_2)/v'_2(x'_2).
\end{aligned}
$$

We know that each of the three points of (x) correspond to (x'_1) and a residual point of (x'). We wish now to prove that the residual point is (x'_2) for each of the three image points of (x). To do this we must prove the following relations:

$$
\begin{aligned}
(4)\quad & u_1(x_1)/u_2(x_1) = u'_1(x'_2)/u'_2(x'_2), \\
& v_1(x_1)/v_2(x_1) = v'_1(x'_2)/v'_2(x'_2), \\
(5)\quad & u_1(x_2)/u_2(x_2) = u'_1(x'_2)/u'_2(x'_2), \\
& v_1(x_2)/v_2(x_2) = v'_1(x'_2)/v'_2(x'_2), \\
(6)\quad & u_1(x_3)/u_2(x_3) = u'_1(x'_2)/u'_2(x'_2), \\
& v_1(x_3)/v_2(x_3) = v'_1(x'_2)/v'_2(x'_2).
\end{aligned}
$$

Relations (4), (5) and (6) are shown to be true by (I′) and (1), (2), (3) respectively. Then (x'_2) is the residual image of each of the three points (x_1), (x_2), (x_3). Also from (4), (5) and (6) the three images of (x'_2) are (x_1), (x_2), (x_3). The point correspondence is therefore a compound involution.

28. *Pencil Cases.*—In accordance with the foregoing theorem, the twelve independent types of general (2, 3) point correspondences* when the equations have but two homogeneous parameters reduce to particular forms of compound involutions. It will be interesting to see to what types of compound involutions they are reduced. (In the following, the Roman numerals refer to types of general (2, 3) point correspondences, the arabic to types of (2, 3) compound involutions.)

The pencil form of Type I is a special case of Type VII for $m = 3$, $n = 2$, which is a particular form of Type 7.

* Hollcroft, loc. cit., page 9.

The pencil form of Type II is a special case of the alternate way of writing Type VII and is therefore a particular case of Type 7.

The pencil form of Type III reduces by quadric inversion to the pencil form of Type I or Type II, depending on the choice of the triangle of inversion, either of which is a special case of Type VII and therefore a particular form of Type 7.

The pencil form of Type IV is a special case of the pencil form of Type VIII for $n = 2$.

The pencil form of Type V goes into the pencil form of Type III by quadric inversion, and is a particular form of Type 7.

The pencil form of Type VI is a special case of Type VII.

The pencil form of Type VII may have its defining equations written as in Type VII each with one less term, or in the alternate form:

(1) $x_1\psi_1' + x_2\psi_2' = 0$

(2) $x_1'\varphi_1 + x_2'\varphi_2 = 0.$

In either case it is a particular form of Type 7.

The pencil form of Type VIII is a particular form of Type 9.

The pencil form of Type IX is a particular form of Type 10.

The pencil form of Type X is transformed into the alternate form of Type VII by quadric inversion and is a particular form of Type 7.

The pencil form of Type XI reduces to a special case of the pencil form of Type III by quadric inversion and is therefore a particular form of Type 7.

The pencil form of Type XII reduces by quadric inversion to a special case of the pencil form of Type IV and is therefore a particular form of Type 9.

29. *Cyclic Cases.*—In Types II, IV and a particular case of Type V of (1, 3) point correspondences, the three image points in the simple plane constitute a cyclic projectivity of period three.* This property of the image points is retained in the (2, 3) compound involutions evolved from these types, for a point of (x') corresponds to one point of (y') thence to one point of (y) and thence to three points of (x). So the three images of a point of (x') are also images of the corresponding point of (y).

Therefore the second, fourth and a particular form of the fifth types of all three classes of (2, 3) compound involutions are such that the three image points in (x) constitute a cyclic projectivity of period three. Furthermore in those types of all three classes, two lines of (x') determine respectively 9, $2n + 1$ and 18 triads of points in (x) each triad constituting a cyclic projectivity of period three.

In all types of (1, 2) point correspondences the two image points of the simple plane are in a simple involution. This same property holds for the

* A. M. Howe, loc. cit., pp. 39–49.

two image points in the triple plane for all (2, 3) compound involutions. Two lines of (x) determine the following numbers of pairs of points in (x'), each pair constituting a cyclic projectivity of period two:

Types 1, 6, 11	16 pairs
Types 2, 7, 12	$2m + 1$ pairs
Types 3, 8, 13	16 pairs
Types 4, 9, 14	36 pairs
Types 5, 10, 15	63 pairs.

30. *Completeness of the Classification.*—It has been shown that the necessary and sufficient condition that a (2, 3) point correspondence be a compound involution is that the image curves of either plane form a net. When this is true the double and triple planes may always be mapped on two other planes by (1, 3) and (1, 2) point correspondences respectively and those two other planes are birationally equivalent. Therefore any (2, 3) compound involution may be obtained by combining a (1, 3) and a (1, 2) involution.

It has been proved that all (1, 2) involutions may be reduced to one of the three independent types herein described.*

By the same method it has been shown that all (1, 3) involutions may be reduced to one of the five independent types described.†

Since all (2, 3) compound involutions can be obtained by combining (1, 2) and (1, 3) involutions, there can be no more independent types of (2, 3) compound involutions than there are possible combinations of the independent types of (1, 2) and (1, 3) involutions. Also since all the (1, 2) and (1, 3) involutions used in these combinations are independent of each other, each combination gives an independent type of (2, 3) compound involutions. There are, then, fifteen independent types of (2, 3) compound involutions and the classification is complete.

WELLS COLLEGE,
March, 1919

* E. Bertini: "Recherche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 8, Vol. 3 (1877), pp. 244–286.

† A. M. Howe, loc. cit., pp. 38–39.

ON SOME PROPERTIES OF GENERAL MANIFOLDS RELATING TO EINSTEIN'S THEORY OF GRAVITATION.

By J. A. Schouten and D. J. Struik.

In two very interesting communications in this Journal* Mr. E. Kasner recently deduced some properties of the four-dimensional manifolds for which $G_{\mu\nu} = 0$. Now it is possible to demonstrate two of these properties in a very short way for n-dimensional manifolds without calculating any three-index-symbol, as a direct consequence of some general theorems. Further it is very easy to give a generalization for manifolds, which obey not Einstein's gravitational equations but the more general equations of de Sitter for a quasi-spherical world.

By V_n is meant a general manifold of n dimensions, by R_n a euclidean one, by S_n a manifold with constant Riemannian curvature and by C_n a manifold which is conformally representable on an R_n.

1. In a recent publication† one of us showed that the necessary and sufficient condition for $n > 3$, that a V_n is a C_n, is that the curvature-tensor (Riemann-Christoffel-Tensor) $B_{\rho\mu\nu\sigma}$ can be written in the form:

$$B_{\rho\mu\nu\sigma} = \frac{1}{n-2}(g_{\rho\nu}L_{\mu\sigma} - g_{\mu\nu}L_{\rho\sigma} - g_{\rho\sigma}L_{\mu\nu} + g_{\mu\sigma}L_{\rho\nu}); \tag{1}$$

$$\rho, \mu, \nu, \sigma = a_1, \cdots, a_n$$

where $L_{\mu\sigma}$ is any symmetrical tensor. From this we conclude

$$G_{\mu\nu} = -L_{\mu\nu} - \frac{1}{n-2}L_{\rho\sigma}g^{\rho\sigma}, \tag{2}$$

and

$$L_{\mu\nu} = -G_{\mu\nu} + \frac{1}{2(n-1)}G_{\rho\sigma}g^{\rho\sigma}. \tag{3}$$

Hence $B_{\rho\mu\nu\sigma}$ vanishes for a C_n if $G_{\mu\nu}$ vanishes:

Of all C_n the only ones for which $G_{\mu\nu} = 0$ are R_n.

For $n = 3$ $B_{\rho\mu\nu\sigma}$ has always the form (1), hence:

Every V_3 for which $G_{\mu\nu} = 0$ is an R_3.

This theorem can be generalized in the following way. If

$$G_{\mu\nu} = \lambda g_{\mu\nu} \tag{4}$$

* Am. Journal of Mathematics, 43 (1921).

† J. A. Schouten, "Ueber die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit kwadratischer Maszbestimmung auf eine Mannigfaltigkeit mit euklidischer Maszbestimmung," *Math. Zeitschr.*, 1921, pp. 58–88.

(in relativity the equation of de Sitter for a quasi-spherical world) from (3) follows

$$L_{\mu\nu} = -\frac{n-2}{2(n-1)}\lambda g_{\mu\nu}, \tag{5}$$

and thus:

$$\begin{aligned} B_{\rho\mu\nu\sigma} &= -\frac{\lambda}{2(n-1)}(g_{\rho\nu}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma} - g_{\rho\sigma}g_{\mu\nu} + g_{\mu\sigma}g_{\rho\nu}) \\ &= -\frac{\lambda}{n-1}(g_{\rho\nu}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma}). \end{aligned} \tag{6}$$

Now it is a well-known theorem that $B_{\rho\mu\nu\sigma}$ being of this form the V_n is an S_n:

Of all C_n the only ones for which $G_{\mu\nu} = \lambda g_{\mu\nu}$ are S_n.
Every V_3 for which $G_{\mu\nu} = \lambda g_{\mu\nu}$ is an S_3.

2. If a V_n is imbedded in a V_{n+1}, and if the indices μ, ν, ρ, σ belong to the fundamental variables in the first manifold, then the generalized theorem of Gauss is:

$$B_{\rho\mu\nu\sigma}' = B_{\rho\mu\nu\sigma} - (h_{\rho\nu}h_{\mu\sigma} - h_{\mu\nu}h_{\rho\sigma}), \tag{7}$$

where $B_{\rho\mu\nu\sigma}$ is the curvature-tensor of the V_n, $B_{\rho\mu\nu\sigma}'$ the curvature-tensor of the V_{n+1} and $h_{\rho\nu}$ the (symmetrical) second fundamental tensor of the V_n with respect to the V_{n+1}.* Now if the V_{n+1} is an R_{n+1}, $B'_{\rho\mu\nu\sigma} = 0$ and from (7) we get:

$$G_{\mu\nu} = g^{\rho\sigma}h_{\mu\rho}h_{\sigma\nu} - h_{\mu\nu}h, \tag{8}$$

where

$$h = h_{\rho\sigma}g^{\rho\sigma}. \tag{9}$$

Now we introduce a system of rectangular coördinates in the principal directions of the symmetrical tensor $h_{\mu\nu}$, which are, according to (8), also the principal directions of $G_{\mu\nu}$. Let the orthogonal components be denoted by the indices $i, j = 1, \cdots, n$. Then $G_{\mu\nu}$ has n (real) components G_{ii}, and $h_{\mu\nu}$ has n (real) components h_{ii}, the components G_{ij}, h_{ij}, $i \neq j$ being zero, and (8) becomes

$$G_{ii} = h_{ii}h_{ii} - h_{ii}\sum_{j}^{1,\ldots,n} h_{jj} \tag{10}$$

where no summations are to be taken that are not indicated by the sign Σ. If now $G_{\mu\nu} = 0$, we get from (10) that either all components h_{ii} vanish or all components h_{ii} but one, in both cases from (7) follows that $B_{\rho\mu\nu\sigma}$ vanishes. Thus:

* See the above cited paper p. 24, or Bianchi-Lukat, Vorlesungenüber Differentialgeometrie, first edition, p. 623, or G. Ricci, Atti Acad. Lincei (1902), p. 359.

No V_n with $G_{\mu\nu} = 0$, that is not an R_n, can be regarded as imbedded in an R_{n+1}. The R_n is either flat or developable.

This theorem can also be generalized. If

$$G_{\mu\nu} = \lambda g_{\mu\nu}, \qquad G_{\mu\nu}' = \lambda' g_{\mu\nu}',^*$$

where $G_{\mu\nu}'$ belongs to the V_{n+1}, then (10) gets the form:

$$\lambda - \lambda' = h_{ii}h_{ii} - h_{ii} \sum_{j}^{1,\ldots,n} h_{jj}.$$

For $n = 2$ h_{11} and h_{22} are not determined by these equations. For $n \neq 2$ it is easy to prove that the only possible solutions are:

$$\begin{aligned} h_{11} = \cdots = h_{mm} &= \pm\sqrt{\frac{m'-1}{m-1}(\lambda - \lambda')}, \\ h_{m+1,\,m+1} = \cdots = h_{nn} &= \mp\sqrt{\frac{m-1}{m'-1}(\lambda - \lambda')}, \\ m' = n - m; \qquad & m = 0, 2, 4 \text{ for } n = 4, \\ & m = 0, 2, 3, \cdots, n-2, n \text{ for } n > 4. \end{aligned}$$

For these manifolds the generalized Codazzi-equations† give the following results:

If for a V_n in C_{n+1} $h_{11} = \cdots = h_{mm}$ is a complete group of equal principal orthogonal components of the second fundamental tensor, then the corresponding congruences of principal curvature form a V_m with only umbilical points with respect to the C_{n+1}. If the C_{n+1} is a S_{n+1} the value of $h_{11} = \cdots = h_{mm}$ on each of the $\infty^{n-m}V_m$ remains constant. If $h_{11} = \cdots = h_{mm}$ remains constant in the V_n, the $\infty^{n-m}V_m$ are geodesic in the V_n.

In consequence of these theorems and the above-mentioned solutions we can prove the theorem:

A V_n with $G_{\mu\nu} = \lambda g_{\mu\nu}$, that is imbedded in an S_{n+1}, $n > 3$, is either a manifold with only umbilical points (a hypersphere) or it contains $\infty^{n-m}V_m$ and also $\infty^m V_{n-m}$ with only umbilical points with respect to the S_{n+1}, where m may have the values $2, \cdots, n-2$. The V_m are geodesic with respect to the V_n and the value of the principal radii of curvature remains constant on the V_n.

Also it can be proved that each of the above-mentioned V_m is imbedded in an S_{m+1} that is geodesic in the S_{n+1} and each of the V_{n-m} in a geodesic S_{n-m+1}.

Hence a V_4 in S_5 with $G_{\mu\nu} = \lambda g_{\mu\nu}$ contains two systems of S_2 with only umbilical points, which have all the same curvature.

* For a V_2 in S_3 these equations are identities.

† Compare Ricci, l.c.

3. The first theorem mentioned under 1 is a special case of a more general theorem.*

If a V_n with fundamental tensor $g_{\mu\nu}$ is represented conformally on a V_n with fundamental tensor $'g_{\mu\nu}$:

$$'g_{\mu\nu} = \sigma^{-1} g_{\mu\nu}, \tag{11}$$

then we can deduce for the curvature-tensors $B_{\rho\mu\nu\sigma}$ and $'B_{\rho\mu\nu\sigma}$ of both V_n:

$$'B_{\rho\mu\nu\sigma} = B_{\rho\mu\nu\sigma} + \tfrac{1}{4}(g_{\rho\nu}p_{\mu\sigma} - g_{\mu\nu}p_{\rho\sigma} - g_{\rho\sigma}p_{\mu\nu} + g_{\mu\sigma}p_{\rho\nu}), \tag{12}$$

where

$$p_{\mu\sigma} = 2s_{\mu\sigma} + s_\mu s_\sigma - \tfrac{1}{2}s_\rho s_\nu g^{\rho\nu} g_{\mu\sigma} \tag{13}$$

$$s_\mu = \frac{1}{\sigma}\frac{\partial\sigma}{\partial x^\mu}.$$

From (11) follows:

$$'G_{\mu\nu} = G_{\mu\nu} + \tfrac{1}{4}(2p_{\mu\nu} - g_{\mu\nu}p - np_{\mu\nu}). \tag{14}$$

Now when $G_{\mu\nu} = 0$ and $'G_{\mu\nu} = 0$ from (14) we get $p_{\mu\nu} = 0$, hence

$$'B_{\rho\mu\nu\sigma} = B_{\rho\mu\nu\sigma}. \tag{15}$$

When two V_n with $G_{\mu\nu} = 0$ are conformally representable on each other the Riemann-Christoffel curvature-tensors are equal.

DELFT, HOLLAND.

* Schouten, ab. cit. p. 79.

GEOMETRICAL THEOREMS ON EINSTEIN'S COSMOLOGICAL EQUATIONS.

BY EDWARD KASNER.

I wish to generalize here some of my results published in the January and April numbers of the AMERICAN JOURNAL OF MATHEMATICS (Vol. 48, 1921, pp. 20, 126), relating to Einstein's original equations of gravitation (in space free from matter),

$$(1) \qquad G_{\mu\nu} = 0.$$

Later Einstein introduced a so-called cosmological term involving a constant λ, the equations being then

$$(2) \qquad G_{\mu\nu} - \lambda g_{\mu\nu} = 0.$$

More recently* he has employed the form

$$(3) \qquad G_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} G = 0,$$

where G is the scalar curvature $g^{\alpha\beta} G_{\alpha\beta}$. I shall refer to (3) simply as the *cosmological equations*. Every solution of the former equations (1) is, of course, a solution of the latter (3), but not vice-versa. The ten equations (3), as Einstein shows, involve one extra dependence as compared with the ten equations (1).

§ 1. FIVE DIMENSIONS.

I shall first take up the question of dimensionality (that is, class of the quadratic form). In the April paper† it was shown that no solution of (1) can represent a 4-spread imbedded in a 5-flat (except in the trivial case where the 4-spread is euclidean, that is, has zero Riemann curvature). We now inquire what solutions of (3) can be imbedded in a 5-flat, and shall find that there are actually two distinct possibilities.

Using the notation of the April paper, we write our spread in the form

$$w = f(x_1, x_2, x_3, x_4).$$

Referring to the formulas on p. 128, we have, in the standard coördinates there employed, involving the four principal curvatures k_i,

$$G_{12} = 0, \text{ etc.}; \qquad G_{11} = -k_1(k_2 + k_3 + k_4), \text{ etc.}$$

* *Berichte Berlin Akad. d. Wiss.* (1919). The complete equations when matter is present of course involve the energy tensor $T_{\mu\nu}$. See also Kopff, *Grundzüge* (1921), pp. 163–165, where the author refers to the field equations of the first, second, third kinds.

† "The Impossibility of Einstein Fields Immersed in Flat Space of Five Dimensions," Vol. 48, pp. 126–129.

Also by the footnote on the same page, we have

$$G = -2(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4).$$

Substituting in the cosmological equations and simplifying, we have the following set of equations for the determination of the four principal curvatures,

$$(4) \quad k_1(k_2+k_3+k_4) = k_2(k_3+k_4+k_1) = k_3(k_4+k_1+k_2) = k_4(k_1+k_2+k_3).$$

Subtracting say the second member from the first, we have

$$(4') \qquad (k_1 - k_2)(k_3 + k_4) = 0,$$

with five similar equations obtained by permuting the subscripts. Hence either

$$(4'') \qquad k_1 = k_2 \text{ or } k_3 = -k_4, \text{ etc.}$$

If the common value of the four expressions in (4) is zero, then we have exactly the system (12) of the April paper, giving merely the trivial case where 3 or 4 of the k's vanish (which means that the manifold is euclidean). Otherwise we find from (4″) two possible types of solution

$$(a) \qquad k_1 = k_2 = \quad k_3 = \quad k_4 \neq 0,$$
$$(b) \qquad k_1 = k_2 = -k_3 = -k_4 \neq 0.$$

In the first case (a) the four principal curvatures are equal at every point, that is, every point is umbilical. It follows then from known theorems that the 4-spread must be a hypersphere. This, of course, checks up since it is known that a 4-dimensional hypersphere is actually imbedded in a 5-flat and is actually a solution of the cosmological equations. (It is sometimes referred to as DeSitter's "Spherical World.")

In the second possibility (b), we have, at every point, the four principal curvatures numerically equal, but two of them are positive and two are negative. We may say then that every point is *semi-umbilical.* It is immediately seen that the Riemann curvature of such a spread is not constant; for we find, in our special coördinates, that the conditions for constant Riemann curvature are

$$(5) \qquad k_1k_2 = k_1k_3 = k_1k_4 = k_2k_3 = k_2k_4 = k_3k_4.$$

For the spherical solutions (a), these products are all equal; but in the new case (b), two of the products are positive and four are negative. We may term a spread of this new type, a *hyperminimal spread,* since we may think of it as a generalization of ordinary minimal surfaces, which have the property that the two principal curvatures are numerically equal, but opposite in sign. The actual existence of such hyperminimal spreads

depends on the consistency of a certain set of three partial differential equations, of the second order, which can readily be written down. It may be that no solution exists, or that all the solutions are imaginary, but the possibility still remains open.

THEOREM I. *If a four-spread imbedded in a five-flat is to obey Einstein's cosmological equations (3), then either every point is umbilical (giving a hypersphere), or else every point is semi-umbilical (giving a possible type of hyperminimal spread).*

§ 2. CONFORMAL REPRESENTATION.

I next take up the generalization of a theorem given in the January paper.* It was there shown that the only solutions of Einstein's original equations (1), which have the same light equation as the euclidean or Minkowski spread, are themselves euclidean; that is, if the spread is to be conformally representable on a 4-flat, it must have zero Riemann curvature. We shall now prove:

THEOREM II. *The only spreads which obey the cosmological equations (3) and which are conformally representable on a 4-flat are those that have constant Riemann curvature (that is, the spread must be of spherical or pseudo-spherical character).*

For this purpose we use the notation of the earlier paper, in particular the formulas on pp. 22–23. Using the value of the tensor $G_{\mu\nu}$ there calculated, we find that the scalar curvature is

$$G = \frac{6}{\lambda}\{\Sigma N_{ii} + \Sigma N_i^2\}; \qquad \text{where} \qquad \lambda = e^{2N}.$$

Substituting then in the equations (3) we find the following system of equations for the determination of the unknown function N:

$$N_{12} - N_1 N_2 = 0, \qquad \text{etc.};$$

$$3N_{11} - N_{22} - N_{33} - N_{44} - 3N_1^2 + N_2^2 + N_3^2 + N_4^2 = 0, \qquad \text{etc.}$$

Employing the same transformation $N = -\log M$ as in the earlier paper, we obtain this simple system:†

$$M_{12} = 0, \qquad M_{13} = 0, \qquad \text{etc.};$$
$$M_{11} = M_{22} = M_{33} = M_{44}.$$

The general solution is obviously

$$(6) \quad M = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5.$$

* "Einstein's Theory of Gravitation: Determination of the Field by Light Signals," Vol. 48, pp. 20–28. Apparently without knowledge of this paper, Ogura has recently given this special theorem in *Comptes Rendus*, Nov. 7, 1921.

† In the January paper, by a typographical error, an extra member $\frac{1}{2}M^{-1}\Sigma M_i^2$ was omitted in the second line of the corresponding set (10), p. 23, but the final result there given is correct.

Our differential form is therefore, since $\lambda = M^{-2}$,

$$ds^2 = M^{-2}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2). \tag{7}$$

This is recognized as a manifold of constant Riemann curvature. (The curvature becomes zero only when the relation $a_1^2+a_2^2+a_3^2+a_4^2-4aa_5=0$ holds, verifying the special theorem of the earlier paper.)

Since a hypersphere can be mapped conformally on a 4-flat, it follows directly from Theorem II that the only cosmological solutions which can be represented conformally on a hypersphere are those of constant Riemann curvature.

We may also generalize the discussion of approximately-euclidean manifolds, given on pp. 27, 28 of the January paper, to approximately-spherical manifolds. The final result is

THEOREM III. *If two approximately-spherical spreads, both obeying the cosmological equations (3), admit conformal representation upon each other (thus having the same light equation), then they are necessarily isometric, except for a homothetic transformation.*

§ 3. SOLUTIONS DEPENDING ON ONE VARIABLE.

A simple example of a solution of (1) where the potentials involve only one of the variables is

$$x_4^{-2}dx_4^2 - x_1^4(dx_1^2 + dx_2^2 + dx_3^2). \tag{8}$$

All orthogonal solutions of this type (see *Bull. Amer. Math. Soc.*, vol. 27 (1920), p. 62) are easily shown to be reducible to the form

$$\begin{gathered} ds^2 = x_1^{2a_1}dx_1^2 + x_1^{2a_2}dx_2^2 + x_1^{2a_3}dx_3^2 + x_1^{2a_4}dx_4^2, \\ a_2 + a_3 + a_4 = a_1 + 1, \qquad a_2^2 + a_3^2 + a_4^2 = (a_1 + 1)^2. \end{gathered} \tag{9}$$

This can be put in the static form, and is completely determined by its light rays.

Analogous solutions of (3) are stated in an abstract printed in *Science* referred to below. We shall state the general result as

THEOREM IV. *All cosmological solutions for which the four potentials in the orthogonal form are functions of one of the four coördinates can be found explicitly by elementary algebraic and transcendental functions. The corresponding spreads can be imbedded in a 7-flat.*

§ 4. AN ALGEBRAIC SOLUTION.

We also state, omitting the easy proof,

THEOREM V. *If the quaternary form $ds^2 = g_{\mu\nu}dx_\mu dx_\nu$ is to be expressible as the sum of two binary forms, one involving say x_1, x_2, the other involving*

say x_3, x_4, and if the cosmological equations (3) are to be obeyed, then the only solution (except for a constant factor) is

$$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

This can be imbedded in a 6-flat with cartesian coördinates $(X_1X_2X_3X_4X_5X_6)$, *the finite representation being*

$$X_1^2 + X_2^2 + X_3^2 = 1, \qquad X_4^2 + X_5^2 + X_6^2 = 1.$$

Excluding the obvious flat and spherical solutions, this is apparently the simplest solution of Einstein's equations which has thus far been obtained, and is the first case where the finite solution is an algebraic spread. In the example (8) given in § 3, the potentials $g_{\mu\nu}$ are algebraic, but not necessarily the corresponding finite spread.

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* The theorems of the present paper were first published in *Science*, Vol. 54 (Sept. 30, 1921), pp. 304–305. A typographical error on page 305 should be corrected as in formula (8) above. See also a paper appearing in the *Mathematischen Annalen*, entitled "The Solar Gravitational Field Completely Determined by its Light Rays." An independent proof of Theorem II, together with very elegant proofs of two of my previous results, applying to forms in n variables, is given in the paper by Prof. J. A. Schouten and Dr. D. J. Struick page 213 of this volume of this JOURNAL. The authors were kind enough to send me a copy of their manuscript.

ON THE FERMAT AND HESSIAN POINTS FOR THE NON-EUCLIDEAN TRIANGLE AND THEIR ANALOGUES FOR THE TETRAHEDRON.

By C. M. Sparrow.

The euclidean triangle has two Fermat or equiangular points. A Fermat point F is defined projectively by the condition that the lines joining F to the absolute point pair I, J are the hessian of the triad of lines joining F to the vertices. If we take actual perpendiculars on the sides as coördinates x_1, x_2, x_3, the points F are transformed by the Desargues transformation $x = 1/y$ into a pair of points H, which are defined most simply by the condition that the feet of the perpendiculars from H on the sides form an equilateral triangle. The Fermat points may be defined in another way which will be found very suggestive. If we consider a particle acted on by three equal forces directed along lines through the vertices, the points F will be positions of equilibrium. We may substitute geometrical for dynamical ideas by considering the potential energy. If r_1, r_2, r_3 are the distances to the vertices the points F give the stationary values of $r_1 \pm r_2 \pm r_3$. The choices of sign, four in number, correspond to the different combinations of pushes and pulls, only two of the four having solutions.

In the non-euclidean plane our point pair I, J is replaced by the conic (line coördinates)

$$\Omega \equiv \xi_1^2 + \xi_2^2 + \xi_3^2 - 2c_1\xi_2\xi_3 - 2c_2\xi_1\xi_3 - 2c_3\xi_1\xi_2 = 0,$$

where the c's are the cosines of the internal angles, which in this case are independent. We will denote by Δ_Ω the discriminant of Ω.

Lines from a point x_1, x_2, x_3 to the vertices are

$$0,\ x_3,\ -x_2; \qquad -x_3,\ 0,\ x_1; \qquad x_2,\ -x_1,\ 0$$

and the hessian pair of this triad is $(1/x_1,\ \omega/x_2,\ \omega^2/x_3)$ and $(1/x_1,\ \omega^2/x_2,\ \omega/x_3)$ where ω is a complex cube root of unity. If x is a Fermat point, this pair touches Ω. The two equations thus obtained reduce to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{2c_3}{x_1x_2} = \frac{1}{x_1^2} + \frac{1}{x_3^2} + \frac{2c_2}{x_1x_3} = \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{2c_1}{x_2x_3}. \qquad (1)$$

Writing $y = 1/x$ we get the pencil of conics

$$y_1^2 + y_2^2 + 2c_3y_1y_2 = y_1^2 + y_3^2 + 2c_2y_1y_3 = y_2^2 + y_3^2 + 2c_1y_2y_3. \qquad (2)$$

The four base points of this pencil satisfy the definition of the hessian

points H. We thus see that in the non-euclidean plane there are four Fermat points and four hessian points. Noting that the equation $x_2^2 + x_3^2 + 2c_1x_2x_3 = 0$ gives the tangents from 1, 0, 0 to Ω and that each of the three conics is on a vertex, we see that each conic is on the vertices of a quadrilateral formed by two pairs of tangents and on the remaining vertex. If $\Delta_\Omega = 0$ two of the points H coincide with I and J and the conics are all circles. The Desargues transformation is in this case isogonal, I and J being interchanged. Thus in the euclidean plane the absolute point pair should in a sense be counted with both sets of points.

We pass now to the tetrahedron, and take as before actual perpendiculars on the faces as coördinates. The equation of the absolute is then

$$\Omega \equiv \Sigma\xi_i^2 + \Sigma 2c_{ij}\xi_i\xi_j = 0, \quad (i, j = 1, 2, 3, 4, \text{ and } i \neq j),$$

where c_{12} is the cosine of the interior angle between the planes 1 and 2, and the c's are all independent for the most general case in non-euclidean space. The definition for the analogue of the Fermat points is suggested by the dynamical conception outlined above. If four equal forces from x to the four vertices are in equilibrium, the bisector of the angle between any pair of forces bisects the angle between the remaining pair, and the two pairs are equally inclined to the bisector. This bisector meets a pair of opposite edges, and the three bisectors are mutually perpendicular. The configuration of four directions may be realized on a sphere by taking a point P and its three reflections in the vertices of a tri-rectangular spherical triangle. If we refer the tetrahedron to these three lines as rectangular cartesian axes it takes the form

$$\pi a, \pi b, \pi c; \qquad \rho a, -\rho b, -\rho c; \qquad -\sigma a, \sigma b, -\sigma c; \qquad -\tau y, -\tau b, \tau c.$$

We are thus led to seek, as the analogue of the Fermat points, those points S such that the lines from S to the pairs of edges are mutually perpendicular. In euclidean space, for which our dynamical conceptions hold, these points are also defined by the condition that $r_1 \pm r_2 \pm r_3 \pm r_4$ shall have stationary values. The more restricted problem of making $r_1 + r_2 + r_3 + r_4$ a minimum has been considered by Sturm,* who notes the orthogonality of the lines from such a point to the edges, but who obtains an analytic solution only as the intersection of three surfaces of the 12th order. Two problems in Wolstenholme's collection may also be noted. The first of these (No. 2024) deals essentially with the dynamical aspect, and the second (No. 2030) with finding points such that the lines to the edges are orthogonal. There is no indication of any connection between the two problems.

* *Crelle J.*, 97, p. 49 (1884).

The three planes

$$1/x_1,\ 1/x_2,\ -1/x_3,\ -1/x_4; \quad 1/x_1,\ -1/x_2,\ 1/x_3,\ -1/x_4;$$
$$1/x_1,\ -1/x_2,\ -1/x_3,\ 1/x_4,$$

each contain two lines on the point x meeting pairs of edges. These planes are mutually perpendicular if

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{2c_{12}}{x_1x_2}=\frac{1}{x_3^2}+\frac{1}{x_4^2}+\frac{2c_{34}}{x_3x_4}, \qquad \text{etc.,} \tag{3}$$

equations which are given by Wolstenholme. If now we write $y = 1/x$ we get the net of quadrics

$$y_1^2+y_2^2+2c_{12}y_1y_2=y_3^2+y_4^2+2c_{34}y_3y_4, \qquad \text{etc.,} \tag{4}$$

defining a set of eight points M* which have the property that the feet of the perpendiculars on the faces form an equifacial tetrahedron. They are thus similar to the hessian points of the triangle, so that the two sets S and M form a quite perfect analogue of the two-dimensional case.

The equations of the net (4) may be rewritten

$$\pi_1y_1=\pi_2y_2=\pi_3y_3=\pi_4y_4,$$

$$\text{where } \pi_i=c_{i1}y_1+c_{i2}y_2+c_{i3}y_3+c_{i4}y_4,\ c_{ii}=1.$$

The points S thus go into the points M by the collineation

$$\pi_1/x_1=\pi_2/x_2=\pi_3/x_3=\pi_4/x_4. \tag{5}$$

The two sets are thus projectively equivalent. The fixed points of the collineation are apolar to (Ω) and to the quadric

$$c_{12}\xi_1\xi_2+c_{13}\xi_1\xi_3+\cdots=0 \tag{6}$$

which touches the faces at the feet of the altitudes. So far no condition has been placed on the c's. If the space is euclidean $\Delta_\Omega = 0$. This condition is equivalent to the coexistence of the equations

$$\begin{aligned}
-A_1+c_{12}A_2+c_{13}A_3+c_{14}A_4&=0,\\
c_{12}A_1-A_2+c_{23}A_3+c_{24}A_4&=0,\\
c_{13}A_1+c_{23}A_2-A_3+c_{34}A_4&=0,\\
c_{14}A_1+c_{24}A_2+c_{34}A_3-A_4&=0,
\end{aligned} \tag{7}$$

where the A's are the areas of the faces. Comparing these with (5) we see that the point and plane A_1, A_2, A_3, A_4 are fixed. The plane is however the plane at infinity, and the point is the "symmedian" point. The

* The discovery of these is due to Dr. F. D. Murnaghan, to whom I owe in its essentials the above elegant treatment, which replaces my own cumbrous solution.

collineation is thus affine, being a pure strain whose axes are the axes of the quadric (6).

The configuration of the eight points has not been in general determined. The equations of the net appear in a canonical form involving 6 constants, which is also the number of absolute invariants of a tetrahedron and quadric. The plane quartic obtained by equating to zero the discriminant of the net also appears in a canonical form, the terms in x^3y, etc., being absent. This would seem to indicate that the net is unrestricted, except when $\Delta_\Omega = 0$. The behavior of the net in special cases throws some doubt on this point, but the problem involves the little known subject of combinants of a net of quadrics, and must be postponed for the present.

Special cases will be considered only briefly. The most important of these is the equifacial tetrahedron (not necessarily euclidean), defined by $c_{14} = c_{23} \equiv c_1$, etc. Writing

$$\varphi_1 = -x_1^2 + x_2^2 + x_3^2 - x_4^2 - 2c_{23}x_2x_3 + 2c_{14}x_1x_4, \text{ etc.}; \; s_1 = \sqrt{1 - c_1^2}, \text{ etc.},$$

the quadrics

$$\varphi_1/s_1 = \pm\, \varphi_2/s_2 = \pm\, \varphi_3/s_3$$

break up into pairs of planes. Four of the points S and M coincide with the incenters $(1, 1, 1, 1)$ $(1, 1, -1, -1)$ $(1, -1, 1, -1)$ $(1, -1, -1, 1)$ and the other four form a tetrahedron in fourfold perspective to this. Further details of this case are omitted. It should be noted however that the equifacial tetrahedron, with 3 absolute invariants, gives rise to a configuration which has none; indicating the possibility that the configuration in the general case may not be that of a general set of "associated" points. Another case that can be completely solved is the "isosceles" tetrahedron

$$c_{12} = c_{13} = c_{23} \equiv c, \qquad c_{14} = c_{24} = c_{34} \equiv c'.$$

The net in this case contains three pairs of planes which all belong to the same pencil of cones, which thus have four common generators.

UNIVERSITY OF VIRGINIA,
March, 1921.

THE CAUCHY-LIPSCHITZ METHOD FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS.*

By William L. Hart.

1. **Introduction.**—The present paper is concerned with the solution of a denumerably infinite system of ordinary differential equations, in which the variables and functions assume real values,

$$(1) \qquad \frac{dx_i}{dt} = f_i(\xi,t) \qquad [i = 1, 2, \cdots; \ \xi = (x_1, \cdots, x_n, \cdots)].$$

Hypotheses will be imposed on (1) under which it will be proved that *at least* one solution exists satisfying given initial conditions.

In a previous paper† the author stated certain theorems concerning completely continuous functions of infinitely many variables, and considered a system (1) in which the f_i were of this type. The unique existence of a solution of (1), satisfying given initial conditions, was established by a generalization of the Picard approximation process. It was found necessary to assume that the f_i satisfied a condition with respect to ξ analogous to the Lipschitz condition used in the consideration of finite systems of differential equations. In the discussion which follows, the system (1) will be considered without the assumption of the Lipschitz condition and, in analogy with the corresponding result for finite systems,‡ it will be established that at least one solution exists satisfying given initial conditions. The proof will be made by use of an extension to infinite systems of the notion of Cauchy polygons, and will be similar in method to that used by Montel§ in establishing the corresponding result for finite systems.

The general system of notation used below is vectorial in character. After any notation has been introduced, it will retain the same meaning as a formal algebraic expression whenever used in the future.

2. **Extension of a Theorem of Arzelà.**—Let R denote the region of points $\xi = (x_1, x_2, \cdots)$, in space of infinitely many dimensions, satisfying the inequalities

$$|x_i - a_i| \leqq r_i \qquad (i = 1, 2, \cdots).$$

It is well known that an analogue of the Weierstrass condensation theorem

* Presented under a different title to the American Mathematical Society, December 28, 1916.

† *Transactions of the American Mathematical Society*, Vol. 18 (1917), p. 125. Referred to in the future as Paper 1.

‡ Cf. P. Montel, *Annales de l'École Normale Supérieure*, 3d series, Vol. 24 (1907), p. 265.

§ Loc. cit., p. 264.

holds in R. That is, if $S = (\xi_n;\ n = 1, 2, \cdots)$ is a sequence of points in R, there exists a point $\xi = (z_1, z_2, \cdots)$ and a sub-sequence (ξ'_n) of S such that

$$\lim_{n=\infty} x'_{in} = z_i \qquad [i = 1, 2, \cdots;\ \xi'_n = (x'_{1n}, x'_{2n}, \cdots)].$$

For a sequence of functions $T = [y_n(t);\ n = 1, 2, \cdots]$, the notion of equal continuity has been defined as follows:*

DEFINITION 1. *The functions of T are equally continuous at a point t_0 if, for every $e > 0$ a number $d > 0$ can be assigned so that on the interval $|t - t_0| \leqq d$ the oscillation of every function of T is at most e.*

If T satisfies Definition 1 at all points t_0 on a closed interval (a, b), it is easily established that the functions of T are equally uniformly continuous on (a, b). In the theorem which follows there is stated, without proof, a property of equally continuous functions recognized originally by Arzelà.†

THEOREM I. *Let the functions of T be defined and equally continuous at every point of an interval P which may be finite or infinite, open or closed. If the maxima of the absolute values of the functions $y_n(t)$, for t on P, possess a common finite bound, then, we may extract a sub-sequence from T which converges at every point of P. Moreover, the convergence is uniform on every closed sub-interval of P.*

Consider a sequence $S' = [\xi'_n(t);\ n = 1, 2, \cdots]$ where, for every t on P, $\xi'_n(t)$ is a point in R. Let the sequence formed by the ith coördinates of the functions of S' be represented by $S_i = [x'_{in}(t);\ n = 1, 2, \cdots]$.

THEOREM II. *For every i suppose that the functions of S_i are equally continuous on P. Then, we may select a sub-sequence $S = [\xi_n(t)]$ of S' and a point $\eta(t)$ belonging to R for every t, such that, for all t on P,*

$$\lim_{n=\infty} x_{in}(t) = y_i(t) \qquad (i = 1, 2, \cdots), \tag{2}$$

where $x_{in}(t)$ and $y_i(t)$ are the i-th coördinates of $\xi_n(t)$ and $\eta(t)$ respectively. Moreover, the convergence in (2) is uniform on every closed sub-interval (a, b) of P.

On applying Theorem I to the sequence S_1, it is seen that we may select a sub-sequence $S^{(1)} = [\xi_n^{(1)}(t)]$ from S' corresponding to which a function $y_1(t)$ exists satisfying

$$\lim_{n=\infty} x_{1n}^{(1)}(t) = y_1(t), \tag{3}$$

uniformly on (a, b), where $x_{1n}^{(1)}(t)$ is the first coördinate of $\xi_n^{(1)}(t)$. As a result of applying Theorem I to the sequence formed by the second coördinates of the functions of $S^{(1)}$, it follows that a sub-sequence

* Cf. Montel, loc. cit., p. 236.

† Cf. Montel, loc. cit., p. 237.

$S^{(2)} = [\xi_n^{(2)}(t)]$ of $S^{(1)}$ and a function $y_2(t)$ may be selected for which there holds

$$\lim_{n=\infty} x_{in}^{(k)}(t) = y_i(t) \qquad (i = 2, k = 2), \tag{4}$$

uniformly for t on (a, b). Obviously, because of (3), condition (4) is also true when $i = 1$. In a similar manner, for every h we obtain a sequence $S^{(h)} = [\xi_n^{(h)}(t)]$ and a function $y_h(t)$ such that the i-th coördinates of $S^{(h)}$ satisfy (4) with ($k = h$; $i = 1, 2, \cdots, h$). It is easily established that a sequence S satisfying Theorem II is obtained by placing $\xi_n(t) = \xi_n^{(n)}(t)$, and by identifying $y_i(t)$ of (2) with that in (4). In view of (2), it is clear that every $y_i(t)$ satisfies $|y_i(t) - a_i| \leqq r_i$.

3. **The Existence Theorem for System** (1).—In equations (1) suppose that (ξ, t) is in the space U defined by the condition (ξ *in* R, $a \leqq t \leqq b$).

DEFINITION* 2. *A function $f(\xi)$, defined in R, is completely continuous at a point ξ in R if, whenever* $\lim_{n=\infty} x_{in} = x_i$ $(i = 1, 2, \cdots)$, *it follows that*

$$\lim_{n=\infty} f(\xi_n) = f(\xi) \qquad [\xi_n = (x_{1n}, x_{2n}, \cdots), \textit{ in } R].$$

It is well known that a function $f(\xi)$, completely continuous at all points of R, possesses a maximum and a minimum, each of which is attained at least once in R. If, for every t, $\xi(t)$ is a point in R whose coördinates $x_i(t)$ are continuous functions of t, it is easily verified that $f[\xi(t)]$ is a continuous function of t. The author has proved† the simple result that, if $(b_1, b_2, \cdots)$ is a sequence of positive numbers and if $e > 0$ is assigned, a number $d > 0$ can be found such that if (ξ', ξ'') are two points of R whose ith coördinates satisfy $|x_i' - x_i''| \leqq db_i$ $(i = 1, 2, \cdots)$, then

$$|f(\xi') - f(\xi'')| \leqq e. \tag{5}$$

In the future let us assume that f_i in (1) satisfies the following conditions:

H_1. If $[(\xi_n, t_n);\ n = 1, 2, \cdots]$ is a sequence of points in U for which

$$\lim_{n=\infty} t_n = t, \qquad \lim_{n=\infty} x_{in} = x_i \qquad (i = 1, 2, \cdots),$$

it follows that $\lim_{n=\infty} f_i(\xi_n, t_n) = f_i(\xi, t)$.

H_2. There exists a number $M > 0$ such that, for every value of i, the maximum m_i of $|f_i(\xi, t)|$ in the region U satisfies $m_i \leqq r_i M$.

THEOREM III. *Suppose that $a \leqq t_0 < b$. Then, there exists at least one function $\xi(t)$ whose coördinates $x_i(t)$ fulfill the initial conditions ($x_i(t_0) = a_i$; $i = 1, 2, \cdots$) and satisfy (1) for $t_0 \leqq t \leqq c$ where c is algebraically the smaller of b and $(t_0 + 1/M)$.*

In order to prove the theorem let us extend the notion of a Cauchy

* Cf. Paper I, p. 129.

† Paper I, p. 130.

polygon,* as defined for a finite system of differential equations, to the case of an infinite system (1).

Let p represent a partition of the values $t \geqq t_0$ with the division points $(t_0, t_1, \cdots, t_m = b)$. Then, the coördinates $b_i(t)$ of the Cauchy polygon $\beta(t)$ for (1), corresponding to the partition p and satisfying $\beta(t_0) = \alpha = (a_1, a_2, \cdots)$, is defined by the following equations, where $i = 1, 2, \cdots$:

$$
(6)\quad \begin{cases}
b_i(t) = a_i + f_i(\alpha, t_0)(t - t_0) & (t_0 \leqq t \leqq t_1), \\
b_i(t) = b_i(t_1) + f_i(\beta_1, t_1)(t - t_1) & [t_1 \leqq t \leqq t_2;\ \beta_1 = \beta(t_1)], \\
\quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\
\quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\
\quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\
b_i(t) = b_i(t_h) + f_i(\beta_h, t_h)(t - t_h) & [t_h \leqq t \leqq t_{h+1};\ \beta_h = \beta(t_h)], \\
\quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\
\quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot \\
\quad\cdot \qquad\qquad \cdot & \qquad\qquad \cdot
\end{cases}
$$

In order to obtain the sequence of polygons which will later be shown to converge to a solution of (1), select a sequence of partitions $(p'_n;\ n = 1, 2, \cdots)$ with norms (d'_n). Suppose that $\lim_{n=\infty} d'_n = 0$. To each partition p'_n there corresponds by (6) a polygon $\beta'_n(t)$. Theorem III will be proved by showing that the sequence $S' = (\beta'_n(t))$ satisfies the hypotheses of Theorem II and by then establishing the fact that the limit function given by that theorem is a solution of (1).

First consider some properties possessed in common by all Cauchy polygons. For every partition p, it follows from (6) and H_2 that, if t is on the interval (t_h, t_{h+1}), where $t_{h+1} \leqq c$, then,

$$
(7)\quad \begin{aligned}
|b_i(t) - a_i| &\leqq |b_i(t) - b_i(t_h)| + |b_i(t_h) - b_i(t_{h-1})| + \cdots + |b_i(t_1) - b_i(t_0)| \\
&\leqq r_i M\{|t - t_h| + |t_h - t_{h-1}| + \cdots + |t_1 - t_0|\} = r_i M(t - t_0) \leqq r_i.
\end{aligned}
$$

Consequently, for all t on (t_0, c), the polygon $\beta(t)$ is defined and in R. In the future suppose that t is on the interval (t_0, c). In the same manner as we obtained (7) it is verified that, for every partition p, and for all points (t', t'') on interval (t_0, c),

$$
(8)\qquad |b_i(t') - b_i(t'')| \leqq M r_i |t' - t''|.
$$

Hence, if $e > 0$ is assigned, a number $w > 0$ can be found so that, for every partition p and for every i,

$$
(9)\qquad |b_i(t') - b_i(t'')| \leqq e r_i,
$$

provided that $|t' - t''| \leqq w$.

Consider the sequence S'. As a consequence of (7) and (9) it follows

* Cf. Bliss, Princeton Colloquium, p. 89.

that S' satisfies the hypotheses of Theorem II on the interval (t_0, c). Let $S = (\beta_n(t))$ be the sub-sequence of S' and let $\xi(t)$ be the point given by Theorem II in the present instance. Equation (2) becomes

$$\lim_{n=\infty} b_{in}(t) = x_i(t) \quad [\beta_n(t) = (b_{1n}(t), b_{2n}(t), \cdots); \; i = 1, 2, \cdots],$$

uniformly for t on (t_0, c). As a consequence of (6) it is obvious that $\xi(t_0) = \alpha$. In view of (9) it is seen that, if $e > 0$ is assigned, a number $w > 0$ may be found such that, if $|t' - t''| \leqq w$, then

$$(10) \qquad |x_i(t') - x_i(t'')| \leqq er_i \qquad (i = 1, 2, \cdots).$$

It remains for us to show that $\xi(t)$ is a solution of (1).

As a consequence of (6) we may state that the coördinates $b_{in}(t)$ of $\beta_n(t)$ satisfy

$$b_{in}(t) = a_i + \int_{t_0}^{t} f_i[\beta_n(\bar{t}), \bar{t}]dt,$$

where $\bar{t}$ is a function of the variable t of integration and is defined as the last division point preceding t in the partition p_n corresponding to $\beta_n(t)$. Let us show that, for every value of i,

$$(11) \qquad \lim_{n=\infty} f_i[\beta_n(\bar{t}), \bar{t}] = f_i[\xi(t), t],$$

uniformly for t on (t_0, c). When this has been proved, we may write

$$x_i(t) = a_i + \int_{t_0}^{t} f_i[\xi(t), t]dt,$$

from which it follows by differentiation that $\xi(t)$ is a solution of (1).

To establish (11) let us consider

$$\begin{aligned} (12) \qquad & |f_i[\beta_n(\bar{t}), \bar{t}] - f_i[\xi(t), t]| \\ & \leqq |f_i[\beta_n(\bar{t}), \bar{t}] - f_i[\beta_n(t), t]| + |f_i[\beta_n(t), t] - f_i[\beta_n(t'), t']| \\ & \quad + |f_i[\beta_n(t'), t'] - f_i[\xi(t'), t']| + |f_i[\xi(t'), t'] - f_i[\xi(t), t]|, \end{aligned}$$

where t' is any point on (t_0, c). In the function $f_i(x_1, x_2, \cdots; t)$ we may think of $(x_1, x_2, \cdots; t)$ as being the coördinates in the space of infinitely many dimensions defined by $(|x_i - a_i| \leqq r_i; \; t_0 \leqq t \leqq c)$. As a consequence of (5), if $e > 0$ is assigned, a number $w > 0$ can be determined so that, if $(|x_i' - x_i''| \leqq wr_i)$ and $|t' - t''| \leqq w$,

$$(13) \qquad |f_i(\xi', t') - f_i(\xi'', t'')| \leqq e.$$

In (12) recall that $|\bar{t} - t| \leqq d_n$, the norm of the partition p_n, and, therefore, as a result of (8),

$$|b_{in}(\bar{t}) - b_{in}(t)| \leqq d_n M r_i.$$

Therefore, it follows from (13) that the first term on the right in (12) approaches zero for $n = \infty$, uniformly for all t on (t_0, c). In view of (9)

and (10), an application of (13) to the second and fourth terms on the right of (12) shows that, if an $e > 0$ is assigned, a number $g > 0$ can be found so that, if $|t - t'| \leqq g$, then each of these terms will be at most $e/3$, for all values of n.

Before considering the third term in (12), let us form a partition ($v_0 = t_0$, $v_1, \cdots, v_m = c$), of the interval (t_0, c), with the number g of the previous paragraph as its norm. Since $f_i(\xi, t)$ is completely continuous and since

$$\lim_{n=\infty} b_{in}(t) = x_i(t),$$

an integer N may be determined so that

$$|f_i[\beta_n(v_j), v_j] - f_i[\xi(v_j), v_j]| \leqq \frac{e}{3} \quad (j = 0, 1, \cdots, m;\ n \geqq N).$$

In (12), for a fixed value of t, suppose that t' is one of the (v_j) differing from t by at most g. Then, if $n \geqq N$, the sum of the last three terms in (12) is at most e. Since N was chosen independently of t, it is seen that the uniformity stated in (11) has been completely established. Hence Theorem III has been proved.

Theorem III was concerned with the existence of a solution on the interval $t \geqq t_0$. A similar thoerem may be stated for the interval $a \leqq t \leqq t_0$.

In paper I the author established* the existence of a unique solution of (1) under the conditions of Theorem III together with an additional assumption. The proof of the uniqueness could be made in the same way in the present paper with the results of Theorem III as a starting point.

UNIVERSITY OF MINNESOTA,
July 1, 1920.

* Loc. cit., p. 144.

BOUNDARY VALUE AND EXPANSION PROBLEMS; FORMULATION OF VARIOUS TRANSCENDENTAL PROBLEMS.*

By R. D. Carmichael.

Proceeding under the guidance afforded by a previous investigation† of the algebraic basis of the theory of certain boundary value and expansion problems we here develop the formal aspects of several transcendental expansion problems which are rather different externally but are abstractly of much the same essence. In each case the work is carried forward to the point of derivation of the form of the expansion and the values of the coefficients; and the problem of determining the nature of the functions having expansions of each of the kinds enumerated is thus clearly set forth. It seemed desirable to present in a single paper this formulation of several problems, so that the reader may most conveniently compare their elements of similarity or difference; and to leave to further investigation the detailed solution of each problem deserving such further treatment. It will be apparent to every one that we have given merely a selection of problems from a much larger number which emerge in a similar way from the consideration of various sorts of limiting cases of the algebraic systems dealt with in the paper already mentioned.

In § 1 we treat the boundary value and expansion problem arising in connection with a single linear differential equation involving a single parameter. In § 2 the corresponding problem for a system of first order equations is more briefly treated and the results are carried over by Volterra's limiting process so as to yield the formulation of a similar expansion problem for certain integro-differential systems; and in § 3 the results of § 2 are rapidly generalized to problems involving ν parameters. Various problems involving difference and integro-difference equations are treated in § 4; these are analogous to the corresponding ones for differential and integro-differential equations developed in §§ 1, 2, 3. In § 5 is given a brief statement of a great variety of mixed problems involving one or more parameters. The character of the problems arising from partial differential equations and integro-differential equations is developed rapidly in § 6 and certain typical cases are briefly treated separately. A particular case of problems arising from partial differential equations is one which is connected with the theory of vibrating plates; this is briefly treated in § 7.

* Presented to the American Mathematical Society, April, 1920.

† American Journal of Mathematics, Vol. XLIII (1921), p. 69.

More details are given in § 1 than elsewhere in the paper, the justification being that those which are omitted are fairly obvious to one who has in mind the corresponding developments in § 1.

1. **The Problem Arising from a Single Differential Equation with One Parameter.**—In connection with the linear differential expressions

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1}u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

$$L_1(u) \equiv g_m \frac{d^m u}{dx^m} + g_{m-1} \frac{d^{m-1}u}{dx^{m-1}} + \cdots + g_1 \frac{du}{dx} + g_0 u, \quad m < n,$$

let us consider also the Lagrange adjoint expressions

$$M(v) \equiv (-1)^n \frac{d^n}{dx^n}[l_n v] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}[l_{n-1}v] + \cdots - \frac{d}{dx}[l_1 v] + l_0 v$$

$$\equiv m_n \frac{d^n v}{dx^n} + \cdots + m_1 \frac{dv}{dx} + m_0 v,$$

$$M_1(v) \equiv h_m \frac{d^m v}{dx^m} + \cdots + h_1 \frac{dv}{dx} + h_0 v.$$

Here it is understood that m may be zero so that $L_1(u)$ may have the particular value $g_0 u$. We assume that the coefficients l_k, g_k, m_k, h_k, with their derivatives of all orders, are real-valued single-valued and continuous functions of x in an interval (ab) and that l_n and g_m do not vanish in (ab). Without loss of generality we take l_n to be positive in (ab). With the linear differential equation

$$(1) \qquad L(u) + \lambda L_1(u) = 0$$

and certain n linear homogeneous conditions in $u(a)$, $u'(a)$, $\cdots$, $u^{(n-1)}(a)$, $u(b)$, $u'(b)$, $\cdots$, $u^{(n-1)}b$,

$$(2) \qquad U_1(u) = 0, \quad U_2(u) = 0, \quad \cdots, \quad U_n(u) = 0,$$

we associate the adjoint linear differential equation

$$(3) \qquad M(v) + \lambda M_1(v) = 0$$

and n like *adjoint conditions*

$$(4) \qquad V_{n+1}(v) = 0, \quad V_{n+2}(v) = 0, \quad \cdots, \quad V_{2n}(v) = 0$$

in such wise as to generalize the corresponding problem treated by Birkhoff* for the case when $L_1(u) \equiv g_0 u$.

We must first make precise the boundary conditions. In doing this as well as in developing the other parts of this section we follow the sugges-

* *Transactions of the American Mathematical Society*, 9 (1908), 373–395; 219–231.

tions afforded by Birkhoff's treatment of the special case. If we write

$$R(u, v) \equiv [l_n v]\frac{d^{n-1}u}{dx^{n-1}} + \left[l_{n-1}v - \frac{d}{dx}(l_n v) \right]\frac{d^{n-2}u}{dx^{n-2}}$$
$$+ \left[l_{n-2}v - \frac{d}{dx}(l_{n-1}v) + \frac{d^2}{dx^2}(l_n v) \right]\frac{d^{n-3}u}{dx^{n-3}} + \cdots$$
$$+ \left[l_1 v - \frac{d}{dx}(l_2 v) + \frac{d^2}{dx^2}(l_3 v) - \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(l_n v) \right] u,$$

$$R_1(u, v)^* \equiv [g_m v]\frac{d^{m-1}u}{dx^{m-1}} + \cdots + \left[g_1 v - \frac{d}{dx}(g_2 v) + \frac{d^2}{dx^2}(g_3 v) - \cdots \right.$$
$$\left. + (-1)^{m-1}\frac{d^{m-1}}{dx^{m-1}}(g_m v) \right] u,$$

then it is well-known and easily verified that we have the identities

$$(5) \quad vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u, v)\}, \qquad vL_1(u) - uM_1(v) \equiv \frac{d}{dx}\{R_1(u, v)\}.$$

Let us now consider the quantity

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b}$$

as a bilinear form in the two sets of variables $u^{(n-1)}(a), u^{(n-2)}(a), \cdots, u'(a), u(a), u^{(n-1)}(b), u^{(n-2)}(b), \cdots, u'(b), u(b)$ and $v(a), v'(a), \cdots, v^{(n-1)}(a), v(b), v'(b), \cdots, v^{(n-1)}(b)$, and let the form be arranged in a square array whose columns contain the u's in the order written and whose rows contain the v's in the order written. The matrix of this bilinear form obviously has the following properties: every element below the main diagonal is zero; every element in the upper right hand fourth of the matrix is zero, the division being made by horizontal and vertical lines through the center; the determinant of the matrix has the non-zero value $\{l_n(a)l_n(b)\}^n$.

Now this bilinear expression may be written in the normal form

$$(6) \qquad [R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v),$$

where the U_i and the V_i are linear and homogeneous functions with constant coefficients, the first in the variables u and the second in the variables v. From the totality of such representations we propose to choose those having certain properties.

In case $L_1(u) \equiv g_0 u$, so that our bilinear form is independent of λ, the linear forms U_i may be chosen in any way so that the $2n$ forms are linearly independent. The forms V_i are then uniquely determined and are linearly independent. This is the choice to be taken in the case of the Birkhoff problem.

* If $m = 0$ we take $R_1(u, v)$ to be identically zero.

In the more general case when the bilinear form contains λ we wish to realize the following conditions: the linear forms $U_1, U_2, \cdots, U_n, V_{n+1}, V_{n+2}, \cdots, V_{2n}$ shall be independent of λ; the whole set $U_1, U_2, \cdots, U_{2n}$ shall be linearly independent for every value of λ; the whole set $V_1, V_2, \cdots, V_{2n}$ shall be linearly independent for every value of λ. For the last two conditions it is necessary and sufficient that the determinant of the set of linear forms U_i shall be independent of λ (except possibly for a factor dependent on λ but vanishing for no value of λ) and different from zero, as one sees readily by aid of the fact that the determinant of the U_i multiplied by the determinant of the V_i has a value equal to the (non-zero) value of the determinant of the bilinear form. That these three conditions may in fact be realized may be seen from the following particular choice of linear forms:

$$U_1 = u^{(n-1)}(a), \quad U_2 = u^{(n-2)}(a), \quad \cdots, U_{n-1} = u'(a), \quad U_n = u^{(n-1)}(b);$$
$$V_{n+1} = v^{(n-1)}(a), \; V_{n+2} = v'(b), \quad V_{n+3} = v''(b), \quad \cdots, \; V_{2n} = v^{(n-1)}(b);$$

for $i \leqq n$, $V_i =$ total factor of U_i in the fundamental bilinear form; for $i > n$, $U_i =$ total factor of V_i in the fundamental bilinear form. Indeed it is now evident that the needful properties may be realized in a great variety of ways.

When the forms U_i and V_i in (6) are determined in accordance with the conditions named, then equation (1) and conditions (2) set a given boundary value problem the *adjoint* of which is set by equation (3) and conditions (4). It is clear, moreover, that problem (1), (2) is likewise the adjoint of problem (3), (4), so that the relation between the two problems is a reciprocal one.

Remark 1.—We have seen that the conditions named can always be realized whatever the value of m, subject merely to the condition that it shall be less than n. For smaller values of m we also have certain additional freedom. We may more generally take k linear forms U_i and $2n - k$ linear forms V_i independent of λ where k is not necessarily equal to n but may be less or greater by an amount depending on m. When m is zero k may be any positive integer less than $2n$.* We may then make corresponding modifications in the two adjoint boundary problems. For the purposes of this paper, however, it is desirable to realize the maximum symmetry in the relations of the two problems; and consequently we confine our attention to the case indicated in the definition as explicitly stated above.

Remark 2.—It is proper to inquire whether the restriction that m shall be less than n is essential. The case $m > n$ reduces to the case $m < n$ by replacing λ by $1/\lambda$ (except for the particular value $\lambda = 0$), so that we

* See a treatment of this case by Bôcher, *Transactions of the American Mathematical Society*, 14 (1913), 403–420.

need to consider only the case $m = n$. In this case the determinant of the bilinear form in the first member of (6) is not in general independent of λ; and hence the properties of the forms U_i and V_i could not be maintained in general, properties which are needed in the further development. These properties might, however, be retained in large measure when $m = n$ provided that g_n and its derivatives have suitable values at a and at b; but we shall not attempt to carry this case along with the other, owing to the special hypotheses which would be necessary and the complications which would raise in certain parts of the work.

Remark 3.—The question of generalization to an equation of the form

$$L(u) + \lambda L_1(u) + \lambda^2 L_2(u) + \cdots + \lambda^k L_k(u) = 0$$

also arises. Under suitable hypotheses it is not difficult to maintain the necessary characteristics of the boundary conditions. Some of the theorems are still maintained without change; but as early as theorem III certain difficulties would arise in general with this more extended case.

We may now readily prove the following theorem:

THEOREM I. *If for $\lambda = \bar{\lambda}$ a solution* $\bar{u}(x)$ of (1), (2) exists, then a solution $\bar{v}(x)$ of (3), (4) also exists for $\lambda = \bar{\lambda}$; if $\bar{u}(x)$ is unique (except for a constant factor), $\bar{v}(x)$ is also unique (except for a constant factor).*

Since $\bar{u}$ is a solution of (1), (2) for $\lambda = \bar{\lambda}$, we have

$$U_1(\bar{u}) = U_2(\bar{u}) = \cdots = U_n(\bar{u}) = 0,$$

and for some k

$$U_{n+k}(\bar{u}) \neq 0,$$

where in U_{n+k} we have the value $\bar{\lambda}$ of λ, since otherwise we should have

$$\bar{u}(a) = \bar{u}'(a) = \cdots = \bar{u}^{(n-1)}(a) = \bar{u}(b) = \bar{u}'(b) = \cdots = \bar{u}^{(n-1)}(b) = 0$$

and our function $\bar{u}(x)$ would be identically equal to zero, contrary to hypothesis.

In the n-fold linear spread of solutions $v(x)$ of (3) for $\lambda = \bar{\lambda}$, there is at least one, say $\bar{v}(x)$, which satisfies the $n - 1$ linear homogeneous conditions

$$V_{n+i}(\bar{v}) = 0,$$

where i runs over all the numbers of the set $1, 2, \cdots, n$ except k. Now we have $L(\bar{u}) + \bar{\lambda} L_1(\bar{u}) = 0$ and $M(\bar{v}) + \bar{\lambda} M_1(\bar{v}) = 0$. Hence if we multiply the second identity in (5) by $\bar{\lambda}$ and add to the first we find on integrating that $R(\bar{u}, \bar{v}) + \bar{\lambda} R_1(\bar{u}, \bar{v})$ is a constant. Therefore from (6) we see that $U_{n+k}(\bar{u}) V_{n+k}(\bar{v}) = 0$; whence it follows that $V_{n+k}(\bar{v}) = 0$, and

* By "solution" we usually mean a function which satisfies the equation and conditions but is not identically equal to zero; when we intend otherwise we shall indicate that fact explicitly.

hence that all the conditions (4) are satisfied by $\bar{v}$, so that it is in fact a solution of (3), (4).

It remains to be shown that $\bar{v}$ is unique whenever $\bar{u}$ is unique. For this purpose let us suppose, if possible, that $\bar{v}$ is not unique. Then two linearly independent solutions of (3), (4) exist, say $\bar{v}$ and v^*. Then k and j exist such that

$$\begin{vmatrix} V_k(\bar{v}) & V_j(\bar{v}) \\ V_k(v^*) & V_j(v^*) \end{vmatrix} \neq 0;$$

for, otherwise, constants c and d would exist,* not both zero, such that $V_l(c\bar{v} + dv^*) = 0$ for $l = 1, 2, \cdots, n$, whence it would follow that the solution $v = c\bar{v} + dv^*$ would satisfy all the conditions $V_i(v) = 0$, $i = 1, 2, \cdots, 2n$, and this is impossible since the V's are linearly independent for every value of λ. Now choose u^*, linearly independent of $\bar{u}$, so as to satisfy the $n - 2$ conditions

$$U_i(u^*) = 0$$

for i running over the numbers of the set $1, 2, \cdots, n$, except k and j. Then from (5) and (6) we see that we have

$$\begin{aligned} U_k(u^*)V_k(\bar{v}) \;+ U_j(u^*)V_j(\bar{v}) \;&= 0, \\ U_k(u^*)V_k(v^*) + U_j(u^*)V_j(v^*) &= 0. \end{aligned}$$

From this we see that $U_k(u^*) = 0 = U_j(u^*)$. Hence u^* satisfies (1), (2). Therefore, if the solution of (3), (4) is not unique for $\lambda = \bar{\lambda}$ neither is the solution of (1), (2) unique. Hence we conclude to the truth of the second part of the theorem.

A value of λ for which the system (1), (2) [and hence the system (3), (4)] has a solution will be called a *characteristic value*. The characteristic value is said to be *simple* if the solution corresponding to it is unique (except for a constant factor).

THEOREM II. *If $y_1, y_2, \cdots, y_n$ are n linearly independent solutions of (1) for $\lambda = \bar{\lambda}$, the condition that $\bar{\lambda}$ is a characteristic value is that the deter-*

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* In more detail one may proceed thus: distinct numbers k_1 and j_1 exist such that

$$V_{k_1}(\bar{v}) \neq 0, \qquad V_{j_1}(v^*) \neq 0.$$

If then

$$V_{k_1}(\bar{v})V_{j_1}(v^*) = V_{k_1}(v^*)V_{j_1}(\bar{v})$$

we have

$$V_{k_1}(v^*) \neq 0, \qquad V_{j_1}(\bar{v}) \neq 0.$$

But if our determinant is always zero it is sufficient to take

$$c = V_{k_1}(v^*), \qquad d = -V_{k_1}(\bar{v}),$$

in order to realize the result stated.

minant

$$\Delta = \begin{vmatrix} U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix}$$

shall vanish; a characteristic value λ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

If we take the general solution of (1) in the form

$$u = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

we see that the vanishing of Δ is a sufficient condition that conditions (2) may be satisfied through an appropriate choice of $c_1, c_2, \cdots, c_n$. When some first minor does not vanish this choice is unique (except for a constant factor).

THEOREM III. *If* $u_i(x)$ *and* $v_j(x)$ *are solutions of (1), (2) and (3), (4), respectively, the first for* $\lambda = \lambda_i$ *and the second for* $\lambda = \lambda_j$, *and if* $\lambda_i \neq \lambda_j$, *then we have*

$$\int_a^b u_iM_1(v_j)dx = 0, \qquad \int_a^b v_jL_1(u_i)dx = 0.$$

From the equations

$$L(u_i) + \lambda_iL_1(u_i) = 0, \qquad M(v_j) + \lambda_jM_1(v_j) = 0$$

we have

$$\{v_j[L(u_i) + \lambda_iL_1(u_i)] - u_i[M(v_j) + \lambda_iM_1(v_j)]\} + (\lambda_i - \lambda_j)u_iM_1(v_j) = 0.$$

If we integrate from a to b we see through aid of (2), (4), (6) that the expression inclosed in braces yields the value zero. Hence we have

$$(\lambda_i - \lambda_j)\int_a^b u_iM_1(v_j)dx = 0.$$

Dividing through by the factor $\lambda_i - \lambda_j$ we have the first relation in the theorem. The second may be proved in a similar way. Moreover, either of these relations follows readily from the other by means of the second relation in (5) and relation (6), looked upon as an identity in λ, together with the boundary conditions.

If we have two sets of functions $u_1(x), u_2(x), u_3(x), \cdots$ and $v_1(x), v_2(x), v_3(x), \cdots$, and the associated adjoint linear forms $L_1(u)$ and $M_1(v)$ such that relations (7) are satisfied for every i and j which are different we shall say that the two sets of functions are *adjoint* each to the other. If the two sets are the same and the form $L_1(u)$ is self-adjoint we shall say that the single set is *self-adjoint*. The conceptions thus introduced are obviously generalizations of the conceptions of biorthogonal sets and orthogonal sets of functions.

If now we have further

$$\int_a^b u_iM_1(v_i)dx \neq 0, \qquad i = 1, 2, 3, \cdots,$$

it is clear that we may formally determine the coefficients c_i in the formal expansion of a given function $f(x)$ in the form

$$f(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots.$$

In fact, if one multiplies by $M_1(v_i)$ and integrates from a to b one has formally

$$c_i = \frac{\int_a^b f(x) M_1(v_i) dx}{\int_a^b u_i M_1(v_i) dx}, \qquad i = 1, 2, 3, \cdots.$$

When λ is not a characteristic number for (1), (2) there exists* a unique $G(x, s; \lambda)$ such that the solution φ of

$$L(\varphi) + \lambda L_1(\varphi) = r, \qquad U_1(\varphi) = U_2(\varphi) = \cdots = U_n(\varphi) = 0,$$

is given by

$$\varphi = \int_a^b G(x, s; \lambda) r(s) ds;$$

and likewise there exists a unique $H(x, s; \lambda)$ such that the solution ψ of

$$M(\psi) + \lambda M_1(\psi) = r, \qquad V_{n+1}(\psi) = V_{n+2}(\psi) = \cdots = V_{2n}(\psi) = 0,$$

is given by

$$\psi = \int_a^b H(x, s; \lambda) r(s) ds.$$

Moreover, we have

$$G(x, s; \lambda) = H(s, x; \lambda).$$

Explicit formulae for G and H are readily obtained (see Bocher, l.c.). The formula for G may be written in terms of a fundamental system $y_1, y_2, \cdots, y_n$ of solutions of (1) as follows:

$$(8) \quad G(x, s; \lambda) = \frac{N(x, s; \lambda)}{\Delta(\lambda)} = \frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & \overline{G}(x, s; \lambda) \\ U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) & U_1(\overline{G}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) & U_n(\overline{G}) \end{vmatrix}}{\Delta(\lambda)}$$

where $\Delta(\lambda)$ has the value given in theorem II and $\overline{G}(x, s; \lambda)$ has the value

$$(9) \quad \overline{G}(x, s; \lambda) = \pm \frac{1}{2} \frac{1}{l_n(s)} \frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}}{\begin{vmatrix} y_1^{(n-1)}(s) & y_2^{(n-1)}(s) & \cdots & y_n^{(n-1)}(s) \\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}},$$

the sign + being taken when $x > s$ and the sign $-$ when $x < s$.

* Bôcher, "Les Méthodes de Sturm," Chapter V.

Now the functions $y_1, y_2, \cdots, y_n$ may be taken analytic in λ. Hence G is analytic in λ except when $\Delta(\lambda) = 0$, that is, except for the characteristic values λ_i of λ for problem (1), (2). If λ_i is a simple characteristic value we may write

$$G(x, s; \lambda) = \frac{R_i(x, s)}{\lambda - \lambda_i} + \sigma(x, s; \lambda) \tag{10}$$

where $\sigma(x, s; \lambda)$ is analytic in λ for $\lambda = \lambda_i$. From (8) we have

$$R_i(x, s) = \frac{N(x, s; \lambda_i)}{\Delta'(\lambda_i)},$$

where $\Delta'(\lambda_i)$ denotes the value at $\lambda = \lambda_i$ of the derivative with respect to λ of $\Delta(\lambda)$. In $N(x, s; \lambda_i)$ the coefficient of $\overline{G}$ is zero. Hence $N(x, s; \lambda_i)$ and its first n derivatives with respect to x are continuous. Moreover, it is easy to see that it satisfies (1), (2); therefore $R_i(x, s)$ is a solution of (1), (2). From the relation between G and H we infer that $R_i(x, s)$ is a solution of (3), (4) in s. From the definition of simple characteristic value it follows then that

$$R_i(x, s) = c_i u_i(x) v_i(s)$$

where c_i is independent of x and s and is yet to be determined.

Now from (10) we have

$$\lim_{\lambda=\lambda_i} \{(\lambda - \lambda_i) G(x, s; \lambda) - c_i u_i(x) v_i(s)\} = 0;$$

whence it follows that

$$\lim_{\lambda=\lambda_i} [(\lambda - \lambda_i) \textstyle\int_a^b G(x, s; \lambda) L_1(u_i(s)) ds - c_i u_i(x) \int_a^b v_i(s) L_1(u_i(s)) ds] = 0. \tag{11}$$

But since

$$L(u_i) + \lambda L_1(u_i) = (\lambda - \lambda_i) L_1(u_i), \quad U_1(u_i) = U_2(u_i) = \cdots = U_n(u_i) = 0,$$

we have from the fundamental property of G that

$$(\lambda - \lambda_i) \int_a^b G(x, s; \lambda) L_1(u_i(s)) ds = u_i(x).$$

Substituting this value into equation (11) we have

$$c_i \int_a^b v_i(s) L_1(u_i(s)) ds = 1.$$

Similarly we should obtain the relation

$$c_i \int_a^b u_i(s) M_1(v_i(s)) ds = 1.$$

Hence we have for the residue $R_i(x, s)$ the value

$$R_i(x, s) = \frac{u_i(x) v_i(s)}{\int_a^b v_i(s) L_1(u_i(s)) ds} = \frac{u_i(x) v_i(s)}{\int_a^b u_i(s) M_1(v_i(s)) ds}. \tag{12}$$

If Γ is a contour in the λ-plane which incloses (just) the characteristic

numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, and if the latter are simple, we have readily

$$(13)\quad \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \int_a^b G(x, s; \lambda) \frac{M_i(v_i(s))}{v_i(s)} f(s) ds d\lambda = \sum_{i=1}^{n} \frac{\int_a^b f(s) M_1(v_i(s)) ds}{\int_a^b u_i(s) M_1(v_i(s)) ds} u_i(x).$$

Thus we have represented in the form of a contour integral the sum of a finite number of terms of the formal expansion

$$(14)\quad f(x) = \sum_{i=1}^{\infty} \frac{\int_a^b f(s) M_1(v_i(s)) ds}{\int_a^b u_i(s) M_1(v_i(s)) ds} u_i(x)$$

of a function $f(x)$ for the case when the characteristic numbers $\lambda_1, \lambda_2, \lambda_3, \cdots$ are all simple and $G(x, s; \lambda)$ has a pole of the first order. At a value λ_i of λ for which these conditions are not satisfied, the corresponding term in the formal expansion in (14) should be replaced by

$$\int_a^b R_i(x, s) \frac{M_1(v_i(s))}{v_i(s)} f(s) ds,$$

where $R_i(x, s)$ denotes the residue of $G(x, s; \lambda)$ at $\lambda = \lambda_i$ and $v_i(s)$ is the solution $N(x, s; \lambda_i)$ of (3), (4) in s. Thus we have a definite formal expansion for every given function $f(x)$ for which the integrals involved exist.

In accordance with the general plan of this paper we rest this investigation at this point. We have carried it far enough to set clearly the fundamental expansion problem associated with it, namely, the problem of determining those functions $f(x)$ for which the given formal expansions are valid. This problem Birkhoff (l.c.) has treated with great success for the special case considered in his paper. It appears likely that the methods of Birkhoff's paper are sufficient to carry through the more general investigation. The results obtained above are sufficient to initiate the further treatment.

In the following sections we derive formal expansions in connection with a great variety of boundary value and expansion problems. We do not go into as much detail as in this section; but the reader who has this section in mind will have no difficulty in seeing how the results actually obtained in each case set the fundamental expansion problem for that case.

2. **The Problem Arising from a System of Differential Equations of the First Order with One Parameter and from Related Integro-differential Equations.**—Let us consider the adjoint systems of differential equations

$$(1)\quad \frac{dy_i}{dx} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij}) y_j, \qquad i = 1, 2, \cdots, n,$$

$$(2)\quad \frac{dz_i}{dx} = \sum_{j=1}^{n} (-a_{ji} - \lambda\alpha_{ji}) z_j, \qquad i = 1, 2, \cdots, n.$$

If for fixed i we multiply these respectively by z_i and y_i, add the resulting equations member by member, sum as to i from 1 to n, and then integrate from a to b, we have

$$[y_1z_1 + y_2z_2 + \cdots + y_nz_n]_{x=a}^{x=b} = 0. \tag{3}$$

Let us now suppose that homogeneous linear boundary conditions (involving the end-points a and b) are set up on the y's and adjoint conditions on the z's so that (3), for the solutions to be considered, is satisfied in virtue of the boundary conditions alone. Moreover, let us suppose that this is done in such a way that the characteristic values of λ for (1) with its conditions are the same as those for (2) with its conditions, the situation being analogous to that treated in § 1, and that the number of characteristic values is infinite. Let us denote the distinct characteristic values and corresponding solutions by the symbols $\lambda^{(k)}$, $y_i^{(k)}$, $z_i^{(k)}$, $k = 1, 2, 3, \cdots$. In (1) let us now replace λ and y_i by $\lambda^{(k)}$ and $y_i^{(k)}$; in (2) let us replace λ and z_i by $\lambda^{(l)}$ and $z_i^{(l)}$, where $\lambda^{(l)} \neq \lambda^{(k)}$; then we have two new systems. For fixed i let us multiply the former by $z_i^{(l)}$ and the latter by $y_i^{(k)}$, add, sum as to i from 1 to n, integrate from a to b, and simplify the first member by means of the equation to which (3) reduces when y_i and z_i are replaced by $y_i^{(k)}$ and $z_i^{(l)}$. Thus we have

$$0 = \int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} [(a_{ij} + \lambda^{(k)}\alpha_{ij})y_j^{(k)}z_i^{(l)} - (a_{ji} + \lambda^{(l)}\alpha_{ji})y_i^{(k)}z_j^{(l)}]dx, \qquad k \neq l;$$

whence, on simplifying and dividing by $\lambda^{(k)} - \lambda^{(l)}$, we have

$$\int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji}y_i^{(k)}z_j^{(l)}dx = 0, \qquad k \neq l. \tag{4}$$

If we suppose that the integral in the first member of (4) is different in value from zero when $k = l$, we have a convenient means for the formal determination of the coefficients c_k (independent of x and i) in the simultaneous formal expansions of n given functions $f_1(x), f_2(x), \cdots, f_n(x)$:

$$f_i(x) = \sum_{k=1}^{\infty} c_k y_i^{(k)}, \qquad i = 1, 2, \cdots, n. \tag{5}$$

For determining these constants c_k we proceed as follows: multiply both members of (5) by $\alpha_{ji}z_j^{(l)}$, sum as to i and as to j from 1 to n, and integrate from a to b; in this way we obtain a relation from which we have readily the value of c_l, namely:

$$c_l = \frac{\int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji}f_iz_j^{(l)}dx}{\int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji}y_i^{(l)}z_j^{(l)}dx}, \qquad l = 1, 2, 3, \ldots. \tag{6}$$

Thus we have in (6) the formulæ for the coefficients c_k, *the same for each function* $f_i(x)$, needed in the joint expansions in (5) of the n functions $f_i(x)$. They are formulæ of great interest.

We have outlined very briefly the foregoing expansion problem, which is probably the same as that of an unpublished paper of Birkhoff (see abstract in *Bulletin of the American Mathematical Society*, (2) 25 (1919), p. 442), because we wish to utilize the suggestion afforded by it for setting up a related problem for certain integro-differential systems obtained from the foregoing differential systems by the limiting process which Volterra has so frequently employed and to such great advantage. It is clear that the notion of *joint expansion* utilized in connection with equation (5) is an essential element in the formulation of the problem. This notion I obtained not from Birkhoff's abstract (which is too brief for details of any sort) but from the dissertation of Dr. C. C. Camp which the author was kind enough to allow me to read in manuscript.* In this dissertation there is a detailed and full treatment of a very special case of the expansion problem formulated above; and the range of ideas developed has been of great use to me in the formulation of the boundary value and expansion problems of this paper.

If we apply to systems (1) and (2) the Volterra limiting process we may obtain the integro-differential equations

$$\frac{\partial y(x, s)}{\partial x} = \int_\alpha^\beta \{u(x, s, t) + \lambda v(x, s, t)\} y(x, t) dt, \tag{7}$$

$$\frac{\partial z(x, s)}{\partial x} = \int_\alpha^\beta \{-u(x, t, s) - \lambda v(x, t, s)\} z(x, t) dt, \tag{8}$$

where the range of variation of x is from a to b while that of s and t is from α to β. Let us multiply both members of (7) by $z(x, s)$ and both members of (8) by $y(x, s)$, add the two resulting equations member by member, and integrate with respect to s from α to β; then, on carrying out certain simplifications through an interchange of the order of integration with respect to s and t and through certain obvious reductions, we have formally the relation

$$\int_\alpha^\beta \frac{\partial}{\partial x} \{y(x, s) z(x, s)\} ds = 0.$$

Integrating with respect to x from a to b, we have

$$\int_\alpha^\beta \{y(b, s) z(b, s) - y(a, s) z(a, s)\} ds = 0, \tag{9}$$

a relation which corresponds to (3) for the differential systems.

* Since this was written a paper by A. Schur has appeared dealing with this problem; see *Mathematische Annalen* 82 (1921): 213–236. It contains (p. 216) a reference to a special case of the problem treated by Hilbert (Göttingen *Nachrichten*, 1906, pp. 474–480). This I had not seen before.

Dr. Camp's article will appear in the next volume of this JOURNAL. [Editor.]

The first member of the last equation is a sort of bilinear form in a non-enumerably infinite set of variables $y(a, s)$, $y(b, s)$, $z(a, s)$, $z(b, s)$, a set of four variables being formed for each s of the interval $(\alpha\beta)$. One desires to have a reduction of this to a normal form analogous to that used in connection with equation (6) of § 1. At present we do not attempt to solve this problem in general. What we desire is to have a set of boundary conditions on y and z separately, linear and homogeneous in character, so that these boundary conditions alone shall imply equation (9) and so that the y-problem and the z-problem shall have the same characteristic values λ infinite in number. Assuming that such equations have been set up, we seek to determine the formal character of the boundary value problem in its simplest aspects.

Let us represent characteristic numbers and corresponding solutions by λ_k, $y_k(x, s)$, $z_k(x, s)$, $k = 1, 2, 3, \cdots$. Let us replace λ and y in (7) by λ_k and y_k; let us replace λ and z in (8) by λ_l and z_l, $k \neq l$; let us multiply the first resulting equation member by member by $z_l(x, s)$ and the second by $y_k(x, s)$, add the two resulting equations member, integrate with respect to s from α to β and with respect to x from a to b, and simplify the resulting relation by aid of what equation (9) becomes when y and z are replaced by y_k and z_l respectively. From both members of the equation which so results remove the non-zero factor $\lambda_k - \lambda_l$; thus we have

$$(10) \qquad \int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s) y_k(x, s) z_l(x, t)\, ds\, dt\, dx = 0, \qquad k \neq l.$$

If we suppose that the first member of (10) is different from zero for $k = l$ and if we seek constants c_k so that a given function $f(x, s)$ shall have the formal expansion

$$(11) \qquad f(x, s) = \sum_{k=1}^{\infty} c_k y_k(x, s)$$

we are led (through the relation obtained on multiplying both members of (11) by $v(x, t, s) z_l(x, t)$ and on integrating appropriately) to the formulæ for c_l, namely:

$$(12) \qquad c_l = \frac{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s) f(x, s) z_l(x, t)\, ds\, dt\, dx}{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s) y_l(x, s) z_l(x, t)\, ds\, dt\, dx}, \qquad l = 1, 2, 3, \cdots.$$

As a special case of the foregoing problem let us take that in which u and v are independent of x and let us write

$$u(x, s, t) = g(s, t), \qquad v(x, s, t) = h(s, t).$$

Solutions exist of the forms

$$y(x, s) = \rho(x)\sigma(s), \qquad z(x, s) = R(x)S(s),$$

where each function in the second members is a function of the single variable indicated. As special boundary conditions implying equation (9) we may take the following:

$$y(b, s) - y(a, s) = 0, \qquad z(b, s) - z(a, s) = 0.$$

These conditions now reduce to the following:

$$\rho(b) - \rho(a) = 0, \qquad R(b) - R(a) = 0.$$

If we substitute the foregoing special values of y and z into the special forms of (7) and (8) indicated we may separate variables in the way which is classic in the study of the partial differential equations of physics and so come through to the following relations:

$$\begin{aligned} \rho'(x) &= \mu\rho(x), \qquad \rho(b) - \rho(a) = 0, \\ \mu\sigma(s) &= \int_a^\beta \{g(s, t) + \lambda h(s, t)\}\sigma(t)dt; \end{aligned} \tag{13}$$

$$\begin{aligned} R'(x) &= -\mu R(x), \qquad R(b) - R(a) = 0, \\ -\mu S(s) &= \int_a^\beta \{g(s, t) + \lambda h(s, t)\}S(t)dt, \end{aligned} \tag{14}$$

where μ is a constant. In (13) and (14) we use the related constants μ and $-\mu$ so as to maintain the requisite property of adjointness. The differential system in each of these pairs of conditions is easily solved, appropriate characteristic values of μ being thus determined. Then the integral equation of the pair is to be solved, appropriate characteristic values of λ having been determined for each value of μ.

Let us further specialize. Suppose that we take $b = \pi$ and $a = -\pi$. Then for μ we have the values $n\sqrt{-1}$, where n is a positive or negative integer, and for $\rho(x)$ and $R(x)$ the corresponding values $e^{nx\sqrt{-1}}$ and $e^{-nx\sqrt{-1}}$ respectively. If we take $g(s, t) \equiv 0$ and employ a non-zero value of μ we have in the second line of (13) and in the second line of (14) an integral equation with parameter λ/μ in the one case and $-\lambda/\mu$ in the other case. It is well-known that the quantities α, β, $h(s, t)$ in this equation may be so chosen that for each value of μ corresponding values of λ exist giving rise to the functions $\cos ms$ for solutions where m runs over the set 1, 2, 3, $\cdots$. For the special case thus defined the expansion (11) has the form

$$f(x, s) = \sum_{m, n=1}^{\infty} (c_{mn} \cos ms e^{nix} + \gamma_{mn} \cos ms e^{-nix}).$$

If $f(x, s)$ is a real-valued function when x and s are real it is clear that c_{mn} and γ_{mn} must be conjugate imaginaries. Then the expansion reduces to the form

$$f(x, s) = \sum_{m, n=1}^{\infty} (a_{mn} \cos nx + b_{mn} \sin nx) \cos ms,$$

where a_{mn} and b_{mn} are real numbers. Furthermore, if $f(x, s)$ is an even function of x as well as of s we must have $b_{mn} = 0$ so that the expansion becomes the double cosine series

$$f(x, s) = \sum_{m, n=1}^{\infty} a_{mn} \cos ms \cos nx.$$

Thus we see that the expansions afforded by (11) and (12) contain as special cases some of the classical expansions of analysis.

The problems formulated in connection with (1), (2) and (7), (8) are capable of various generalizations. It will be sufficient to indicate briefly one of them. Let us consider the system of equations

$$(15)\quad \frac{\partial y_i(x, s)}{\partial x} = \sum_{j=1}^{n} (a_{ij} + \lambda \alpha_{ij}) y_j(x, s) + \int_\alpha^\beta \sum_{j=1}^{n} [\rho_{ij}(x, s, t) + \lambda \sigma_{ij}(x, s, t)] y_j(x, t) dt,$$

$$(16)\quad \frac{\partial z_i(x, s)}{\partial x} = \sum_{j=1}^{n} (- a_{ji} - \lambda \alpha_{ji}) z_j(x, s) + \int_\alpha^\beta \sum_{j=1}^{n} [- \rho_{ji}(x, t, s) - \lambda \sigma_{ji}(x, t, s)] z_j(x, t) dt,$$

for $i = 1, 2, \cdots, n$. Here in general the a_{ij} and α_{ij} are functions of x and s. In the special case when $\rho_{ij} \equiv 0 \equiv \sigma_{ij}$ and a_{ij} and α_{ij} are functions of x alone, we may take y_i and z_i to be functions of x alone, and we have our systems (1) and (2). It is clear that equations (7) and (8) are included in another way as a special case.

Let us multiply (15) by $z_i(x, s)$ and (16) by $y_i(x, s)$, integrate the resulting equations as to s from α to β and in the relation coming from (16) interchange the order of double integration and in the part of the formula containing this double integration interchange s and t, add the two equations thus gotten from (15) and (16), and in the resulting equation sum as to i from 1 to n. Finally in this resulting equation integrate as to x from a to b. Then we have the relation

$$(17)\quad \int_\alpha^\beta [\sum_{i=1}^{n} \{y_i(b, s) z_i(b, s) - y_i(a, s) z_i(a, s)\}] ds = 0.$$

This corresponds to our previous relation (9).

Let us suppose that linear homogeneous boundary conditions on the y_i and the z_i separately are so chosen that relation (17) is implied by these boundary conditions, that the y-problem and the z-problem have the same characteristic values λ infinite in number, and let us denote the corresponding values and solutions by λ_k, $y_i^{(k)}$, $z_i^{(k)}$, $k = 1, 2, 3, \cdots$. Then by a

method of procedure which is now obvious we come through to the fundamental reciprocal relation

$$(18)\quad \begin{aligned} &\int_a^b\{\int_\alpha^\beta[\sum_{i=1}^n\sum_{j=1}^n\alpha_{ji}(x,s)y_i^{(k)}(x,s)z_j^{(l)}(x,s)]ds \\ &\quad + \int_\alpha^\beta\int_\alpha^\beta[\sum_{i=1}^n\sum_{j=1}^n\sigma_{ji}(x,t,s)y_i^{(k)}(x,s)z_j^{(l)}(x,t)]dsdt\}dx = 0, \quad k \neq l. \end{aligned}$$

This is the generalized biorthogonality condition of which we have already found special cases.

Let us denote by D_{kl} the first member of (18) and let us suppose that $D_{kk} \neq 0$ for each k. Then if we have a set of expansions of the form

$$(19)\qquad f_i(x, s) = \sum_{r=1}^{\infty} c_r y_i^{(r)}(x, s), \qquad i = 1, 2, \cdots, n,$$

giving the simultaneous expansions of the n functions $f_i(x, s)$, $i = 1, 2, \cdots, n$, in terms of the $y_i^{(r)}(x, s)$, the coefficients c_r may be found by multiplying both sides of the equation

$$\sum_{j=1}^n \alpha_{ji}(x, s)z_j^{(\nu)}(x, s) + \int_\alpha^\beta[\sum_{j=1}^n\sigma_{ji}(x, t, s)z_j^{(\nu)}(x, t)]dt,$$

summing as to i from 1 to n, integrating as to s from α to β and as to x from a to b, and simplifying by use of (18); thus we have

$$(20)\quad \begin{aligned} D_{\nu\nu}c_\nu &= \int_a^b\{\int_\alpha^\beta[\sum_{i=1}^n\sum_{j=1}^n\alpha_{ji}(x, s)f_i(x, s)z_j^{(\nu)}(x, s) \\ &\quad + \int_\alpha^\beta\sum_{i=1}^n\sum_{j=1}^n\sigma_{ji}(x, t, s)z_j^{(\nu)}(x, t)f_i(x, s)dt]ds\}dx, \quad \nu = 1, 2, 3, \cdots. \end{aligned}$$

It is clear that one may apply to (15) and (16) the Volterra limiting process and so replace these equations by others in which the discrete variable i has given way to an additional continuous variable, the resulting integro-differential equation containing a repeated integration, and that the formal expansion problem subsists in the new form. One might then employ a system of such equations and apply again to them the Volterra limiting process; and so on indefinitely.

3. **Differential and Integro-differential Problems Involving ν Parameters.**—Let us consider the ν systems of equations

$$(1)\quad \frac{dy_{ki}}{dx} = \sum_{j=1}^{n_k}(a_{0kij} + \lambda_1 a_{1kij} + \lambda_2 a_{2kij} + \cdots + \lambda_\nu a_{\nu kij})y_{kj}, \qquad i = 1, 2, \cdots, n_k,$$

a separate system being formed for each value k of the set $1, 2, 3, \cdots, \nu$; and with these let us associate the ν systems of adjoint equations

$$(2)\quad \frac{dz_{ki}}{dx} = \sum_{j=1}^{n_k}(-a_{0kji} - \lambda_1 a_{1kji} - \cdots - \lambda_\nu a_{\nu kji})z_{kj}, \qquad i = 1, 2, \cdots, n_k.$$

For fixed k and i multiply equation (1) member by member by z_{ki} and equation (2) member by member by y_{ki}, add the two resulting equations member by member, sum as to i from 1 to n_k, and integrate with respect to x from a to b; thus we have

$$(3) \qquad \left[y_{k1}z_{k1} + y_{k2}z_{k2} + \cdots + y_{kn_k}z_{kn_k}\right]_{x=a}^{x=b} = 0, \qquad k = 1, 2, \cdots, \nu.$$

Let us now suppose that for each value of k we set up on the quantities $y_{ki}(b)$, $y_{ki}(a)$, $i = 1, 2, \cdots, n_k$, and separately on the quantities $z_{ki}(b)$, $z_{ki}(a)$, $i = 1, 2, \cdots, n_k$, linear homogeneous boundary conditions (analogous to those developed in detail in § 1) of such sort that relations (3) are implied by the boundary conditions alone and that the y-problem and the z-problem have the same sets $\lambda_1^{(s)}, \lambda_2^{(s)}, \cdots, \lambda_\nu^{(s)}$ of characteristic values, that the number of sets is enumerably infinite and that these distinct sets are denoted by the named symbols for $s = 1, 2, 3, \cdots$. Let the corresponding solutions be denoted by $y_{ki}^{(s)}$, $z_{ki}^{(s)}$.

In equation (1) let us replace the quantities y and λ by corresponding quantities $y^{(s)}$ and $\lambda^{(s)}$; in equation (2) let us replace the quantities z and λ by corresponding quantities $z^{(r)}$ and $\lambda^{(r)}$ where $r \neq s$. For a fixed k multiply the first resulting equation member by member by $z_{ki}^{(r)}$ and the second by $y_{ki}^{(s)}$, add the two resulting equations member by member, sum as to i from 1 to n_k, integrate with respect to x from a to b, and simplify by what (3) becomes on replacing the quantities y and z by the corresponding quantities $y^{(s)}$ and $z^{(r)}$. In this way we are led to the following relations:

$$(4) \quad (\lambda_1^{(s)} - \lambda_1^{(r)})\int_a^b \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} a_{1kji}y_{ki}^{(s)}z_{kj}^{(r)}dx + \cdots$$
$$+ (\lambda_\nu^{(s)} - \lambda_\nu^{(r)})\int_a^b \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} a_{\nu kji}y_{ki}^{(s)}z_{kj}^{(r)}dx = 0,$$

a separate relation being formed for each value k of the set $1, 2, \cdots, \nu$. Now the differences $\lambda_1^{(s)} - \lambda_1^{(r)}, \cdots, \lambda_\nu^{(s)} - \lambda_\nu^{(r)}$ are not simultaneously zero; hence the determinant of their coefficients in the system (4) must have the value zero, that is, we must have

$$\left|\int_a^b \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} a_{lkji}y_{ki}^{(s)}z_{kj}^{(r)}dx\right| = 0, \qquad r \neq s,$$

where the term of the determinant explicitly written is that in the lth column and kth row.

The last relation may be reduced to a more convenient form. In the kth row of the determinant let us replace the variable x of integration by t_k and the variables i and j of summation by i_k and j_k respectively. Then the relation reduces to

$$\int_a^b \int_a^b \cdots \int_a^b \left|\sum_{i_k=1}^{n_k}\sum_{j_k=1}^{n_k} a_{lkj_ki_k}(t_k)y_{ki_k}^{(s)}(t_k)z_{kj_k}^{(r)}(t_k)\right| dt_1dt_2 \cdots dt_\nu = 0, \qquad r \neq s;$$

the determinant employed being that whose element in lth column and kth row is that explicitly written. Then if we employ the symbol D to represent the determinant

(5) $$D_{i_1j_1 \cdots i_\nu j_\nu}(t_1, t_2, \cdots, t_\nu) = |a_{lkj_ki_k}(t_k)|$$

whose element in lth column and kth row is that explicitly written, we have for the final form of our generalized biorthogonality conditions the following:

(6) $$\int_a^b\int_a^b\cdots\int_a^b\left\{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_\nu=1}^{n_\nu}\sum_{j_\nu=1}^{n_\nu} D_{i_1j_1\cdots i_\nu j_\nu}(t_1, t_2, \cdots, t_\nu)\right.$$
$$\left.\times \prod_{h=1}^{\nu} y_{hi_h}^{(s)}(t_h) z_{hj_h}^{(r)}(t_k)\right\} dt_1dt_2 \cdots dt_\nu = 0, \qquad r \neq s.$$

In what follows we shall suppose that we have before us the case in which the first member of (6) is different from zero when $r = s$.

Now let us consider the set of functions

$$f_{i_1i_2\cdots i_\nu}(t_1, t_2, \cdots, t_\nu), \qquad i_k = 1, 2, \cdots, n_k \text{ for } k = 1, 2, \cdots, \nu,$$

with regard to the question of their simultaneous expansion in the form

(7) $$f_{i_1i_2\cdots i_\nu}(t_1, t_2, \cdots, t_\nu) = \sum_{s=1}^{\infty} c_s y_{1i_1}^{(s)}(t_1) y_{2i_2}^{(s)}(t_2) \cdots y_{\nu i_\nu}^{(s)}(t_\nu),$$

the coefficients c_s being the same for every function of the set. In case such an expansion exists we have a ready means of determining the coefficients c_s. The form of the first member of (6) suggests the operations to be performed. Let us denote by B_{rs} the first member of (6) and let us suppose that $B_{rr} \neq 0$ for $r = 1, 2, 3, \cdots$. Then for the determination of the c_r we have the relations

(8) $$B_{rr}c_r = \int_a^b\int_a^b\cdots\int_a^b\left\{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\right.$$
$$\cdots\sum_{i_\nu=1}^{n_\nu}\sum_{j_1=1}^{n_\nu} D_{i_1j_1\cdots i_\nu j_\nu}(t_1, t_2, \cdots, t_\nu) f_{i_1i_2\cdots i_\nu}(t_1, t_2, \cdots, t_\nu)$$
$$\left.\cdot z_{1i_1}^{(r)}(t_1) z_{2i_2}^{(r)}(t_2) \cdots z_{\nu i_\nu}^{(r)}(t_\nu)\right\} dt_1dt_2 \cdots dt_\nu, \qquad r = 1, 2, 3, \cdots.$$

The expansion problem which is set by these formal results we may look upon as a generalization of that briefly treated at the beginning of § 2. From the fuller treatment of a related problem given in § 1 one sees readily how to proceed to a more precise formulation of the problem here briefly characterized. Moreover, it is clear that one may formulate the problem at once for a system of ν equations of general orders and so include in one form (rather complicated, to be sure) the three differential equations problems already set.

We saw in § 12 how the integro-differential equations problem (7), (8)

grew out of the differential equations problem (1), (2). In a similar way the problem (1), (2) of this section gives rise to a new problem associated with the equations

$$(9)\qquad \frac{\partial y_k(x, s)}{\partial x} = \int_\alpha^\beta \{a_{0k}(x, s, t) + \lambda_1 a_{1k}(x, s, t) + \cdots + \lambda_\nu a_{\nu k}(x, s, t)\} y_k(x, t)dt,$$

$$(10)\qquad \frac{\partial z_k(x, s)}{\partial x} = \int_\alpha^\beta \{- a_{0k}(x, t, s) - \lambda_1 a_{1k}(x, t, s) - \cdots - \lambda_\nu a_{\nu k}(x, t, s)\} z_k(x, t)dt.$$

With the guide afforded by the earlier part of this section and the special case of our present problem treated in § 2 one may proceed readily to set up the generalized biorthogonality conditions and by aid of them to determine the coefficients in the formal expansion of a function of 2ν variables.

Corresponding to equation (6) we should thus have

$$(11)\qquad \int_a^b\int_a^b\cdots\int_a^b\{\int_\alpha^\beta\int_\alpha^\beta\cdots\int_\alpha^\beta D(x_1, x_2, \cdots, x_\nu, s_1, t_1, s_2, t_2, \cdots, s_\nu, t_\nu) \cdot\prod_{h=1}^{\nu} y_h^{(s)}(x_h, s_h) z_h^{(r)}(x_h, t_h)ds_1dt_1 \cdots ds_\nu dt_\nu\}dx_1dx_2 \cdots dx_\nu = 0, \qquad r \neq s,$$

where the symbol D denotes the determinant of order ν whose element in lth column and kth row is $a_{lk}(x_k, s_k, t_k)$. If we denote the first member of this relation by C_{rs} and suppose that $C_{rr} \neq 0$ for every r, then for the coefficients c_r in the expansion

$$(12)\qquad f(x_1, x_2, \cdots, x_\nu, s_1, s_2, \cdots, s_\nu) = \sum_{s=1}^{\infty} c_s \prod_{h=1}^{\nu} y_h^{(s)}(x_h, s_h)$$

we have formally

$$(13)\qquad C_{rr}c_r = \int_a^b\int_a^b\cdots\int_a^b\{\int_\alpha^\beta\int_\alpha^\beta\cdots\int_\alpha^\beta D(x_1, \cdots, x_\nu, s_1, t_1, \cdots, s_\nu, t_\nu) f(x_1, \cdots, x_\nu, s_1, \cdots, s_\nu) \cdot\prod_{h=1}^{\nu} z_h^{(r)}(x_h, t_h)\cdot ds_1dt_1 \cdots ds_\nu dt_\nu\} \times dx_1dx_2 \cdots dx_\nu, \qquad r = 1, 2, 3, \cdots.$$

These results might, if it were desired, be also extended by means of the suggestion afforded by the problem associated with equations (15) and (16) of § 2 and the generalizations of it which are mentioned there.

4. **Problems Involving Difference and Integro-difference Equations.**—Let us consider the adjoint systems of difference equations

$$(1)\qquad u_i(x + 1) = \sum_{j=1}^{n}(\delta_{ij} + \varphi_{ij} + \lambda\psi_{ij})u_j(x), \qquad i = 1, 2, \cdots, n,$$

$$(2)\qquad v_i(x) = \sum_{j=1}^{n}(\delta_{ji} + \varphi_{ji} + \lambda\psi_{ji})v_j(x + 1), \quad i = 1, 2, \cdots, n,$$

where δ_{ij} has the value 1 or 0 according as i is or is not equal to j and φ_{ij}

and ψ_{ij} are functions of x which are analytic at infinity and vanish there to an order at least as high as the second. Under such hypotheses these equations have for a fixed value of λ fundamental systems*

$$u_{1j}(x),\ u_{2j}(x),\ \cdots,\ u_{nj}(x);\ v_{1j}(x),\ v_{2j}(x),\ \cdots,\ v_{nj}(x)$$

of solutions each function of which is analytic throughout the finite plane except for a set of singularities in a left half-plane; moreover, if x approaches infinity along any ray from the origin exclusive of the negative axis of reals or along any line proceeding to the right parallel to the axis of reals, we have

$$\lim u_{ij}(x) = \delta_{ij}, \qquad \lim v_{ij}(x) = \delta_{ij}.$$

Any solution of either equation formed from the foregoing solutions of that equation by taking a linear combination of such solutions with constant coefficients will have the property that each function in it approaches a limiting value as x approaches infinity in the manner specified. We confine attention to such solutions of equations (1) and (2).

Now if we multiply equation (1) member by member by $v_i(x+1)$ and (2) by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n}\Delta\{u_i(x)v_i(x)\} = 0. \tag{3}$$

If the real part of a is sufficiently large we have u_i and v_i analytic at every point x whose real part is not less than the real part of a. Hence in (3) we may sum as to x from a to infinity, where x runs over the values a, $a+1$, $a+2$, $\cdots$; thus we have

$$\sum_{i=1}^{n}\{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0. \tag{4}$$

This is clearly analogous to equation (3) of § 2 and differs from it (apart from notation) only in having the limit ∞ instead of the limit b.

Let us now suppose that adjoint homogeneous linear boundary conditions are set up on the $u_i(\infty)$, $u_i(a)$ on the one hand and on the $v_i(\infty)$, $v_i(a)$ on the other hand so that relation (4) is implied by the boundary conditions and that this is done in such a way that the characteristic values of λ for (1) and the conditions on the u_i are the same as those for (2) and the conditions on the v_i and that the number of characteristic values is infinite. Let us denote the distinct characteristic values and corresponding solutions by $\lambda^{(k)}$, $u_i^{(k)}$, $v_i^{(k)}$, $k = 1, 2, 3, \cdots$.

In (1) let us now replace λ and u_i by $\lambda^{(k)}$ and $u_i^{(k)}$, in (2) let us replace λ and v_i by $\lambda^{(l)}$ and $v_i^{(l)}$, where $l \neq k$; then we have two new systems. Let

* See AMERICAN JOURNAL OF MATHEMATICS, 35 (1913), 161–162.

us multiply the first resulting equation member by member by $v_i^{(l)}(x+1)$ and the second by $-u_i^{(k)}(x)$, add the resulting equations member by member, sum as to i from 1 to n, sum as to x over the set $a, a+1, a+2, \cdots$, simplify by aid of what (4) becomes when u_i and v_i are replaced by $u_i^{(k)}$ and $v_i^{(l)}$ respectively, and divide by the non-zero quantity $\lambda^{(k)} - \lambda^{(l)}$. Thus we have

$$(5) \qquad \sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(a+t)u_i^{(k)}(a+t)v_j^{(l)}(a+1+t) = 0, \quad k \neq l.$$

If we suppose that the quantity in the first member of (5) is (for all k) different from zero when l is replaced by k we shall have a ready means for the formal determination of the coefficients c_k in the (assumed) expansions

$$(6) \qquad f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \quad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the n functions. For this purpose we multiply both sides of equation (6) by $\psi_{ji}(x)v_j^{(k)}(x+1)$, sum as to i and j each from 1 to n, and sum as to x over the set a, $a+1$, $a+2, \cdots$; employing (5) we come through readily to the relations

$$(7) \qquad c_k = \frac{\displaystyle\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(a+t)f_i(a+t)v_j^{(k)}(a+1+t)}{\displaystyle\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(a+t)u_i^{(k)}(a+t)v_j^{(k)}(a+1+t)}, \qquad k = 1, 2, 3, \cdots.$$

Since the coefficients c_k are determined merely by the function-values at a discrete set of values of x it may seem at first sight that this is not likely to lead to an expansion theory of much interest. But it would be natural to confine attention to functions $f_i(x)$ having suitably restricted properties, especially in the neighborhood of infinity, and for such functions it may be anticipated that the theory is valuable. Moreover, it furnishes also a natural means of interpolation of a function defined initially only over the set of points a, $a+1$, $a+2$, $\cdots$, the function values at these points being the only ones needed in the determination of the coefficients c_k. Again, it affords a natural means of formal expansion of functions defined only for positive integral values of the argument, such as is the case with the usual number-theoretic functions.

Let us rapidly illustrate the foregoing theory by considering the problem associated with the equations

$$(8) \qquad u(x+1) - \left(1 - \frac{\mu^2}{x^2}\right)u(x) = 0,$$

$$(9) \qquad \left(1 - \frac{\mu^2}{x^2}\right)v(x+1) - v(x) = 0,$$

where μ^2 is the parameter. These equations have the solutions

$$(10) \qquad u(x) = \frac{\Gamma(x-\mu)\Gamma(x+\mu)}{[\Gamma(x)]^2}, \qquad v(x) = \frac{[\Gamma(x)]^2}{\Gamma(x-\mu)\Gamma(x+\mu)},$$

where $\Gamma(x)$ denotes the usual gamma function; and the general solution for each case is gotten by multiplying the particular solution by an arbitrary periodic function of period unity. But if $u(x)[v(x)]$ is to approach a definite limiting value when x approaches infinity in the sense defined above this periodic multiplier must reduce to a constant; and such constant multiplier is clearly irrelevant to the problem in hand. Hence, for our purposes, we take the solutions given in (10).

Now condition (4) reduces in the present case to $u(\infty)v(\infty) - u(a)v(a) = 0$. As suitable boundary conditions implying this relation we may take

$$(11) \qquad t\,u(\infty) - u(a) = 0, \qquad \frac{1}{t}v(\infty) - v(a) = 0,$$

where t is a given non-zero constant. Now from (10) and the asymptotic properties of the gamma function we see that $u(\infty) = 1 = v(\infty)$. Then the condition of consistency for the u-problem and that for the v-problem both reduce to the following:

$$(12) \qquad \Gamma(a-\mu)\Gamma(a+\mu) = t\{\Gamma(a)\}^2.$$

The roots of this equation give the common characteristic values of μ^2 for the u-problem and the v-problem. [For each solution μ of (12) there is also the solution $-\mu$; but only their common square μ^2 is a characteristic value for our problem.]

For varying choice of a and t the expansion problem which results brings a variety of properties to notice. One of the most interesting cases is that in which $a = \frac{1}{2}$, $-1 \leqq 1/t \leqq +1$; in this case equation (12) readily becomes

$$(13) \qquad \cos \pi \mu = \frac{1}{t},$$

if one reduces by aid of the relation $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$. Then the characteristic values μ^2 are readily determined in terms of a single one of them. Let $\bar{\mu}$ be the smallest positive value of μ satisfying (13); then the characteristic values μ^2 are

$$\mu^2 = (\bar{\mu} + 2n)^2, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

Corresponding to these let us denote the solutions by $u_n(x)$, $v_n(x)$ so that

we have*

$$(14)\qquad u_n(x) = \frac{\Gamma(x-\bar{\mu}-2n)\Gamma(x+\bar{\mu}+2n)}{[\Gamma(x)]^2}, \qquad v_n(x) = \frac{1}{u_n(x)},$$
$$n = 0, 1, 2, \cdots.$$

The biorthogonality conditions now reduce to

$$(15)\qquad \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^2} \frac{u_n(k+\frac{1}{2})}{u_m(k+\frac{3}{2})} = 0, \qquad m \neq n.$$

Now we may apply to systems (1) and (2) a Volterra limiting process analogous to that by which we arrived at equations (7) and (8) in § 2. By so doing we are led to the integro-difference equations †

$$(16)\qquad \Delta u(x, s) = \int_a^\beta \{\varphi(x, s, t) + \lambda\psi(x, s, t)\} u(x, t)dt,$$
$$(17)\qquad -\Delta v(x, s) = \int_a^\beta \{\varphi(x, t, s) + \lambda\psi(x, t, s)\} v(x+1, t)dt.$$

Corresponding to equations (4) to (7) we now have the following:

$$(18)\qquad \int_a^\beta \{u(\infty, s)v(\infty, s) - u(a, s)v(a, s)\} ds = 0,$$

$$(19)\qquad \sum_{r=0}^{\infty} \int_a^\beta \int_a^\beta \psi(a+r, t, s)u_k(a+r, s)v_l(a+1+r, t)dsdt = 0, \qquad k \neq l,$$

$$(20)\qquad f(x, s) = \sum_{k=1}^{\infty} c_k u_k(x, s),$$

$$(21)\qquad c_k = \frac{\sum_{r=0}^{\infty} \int_a^\beta \int_a^\beta \psi(a+r, t, s)f(a+r, s)v_k(a+1+r, t)dsdt}{\sum_{r=0}^{\infty} \int_a^\beta \int_a^\beta \psi(a+r, t, s)u_k(a+r, s)v_k(a+1+r, t)dsdt},$$
$$k = 1, 2, 3, \cdots.$$

The foregoing work of this section is analogous to a part of that of § 2. It is clear that the analogues of the other parts of § 2 exist here and can be readily developed. In fact, we can develop the theory of difference and integro-difference equations in forms analogous to all parts of the problems treated or suggested in §§ 1 to 3 for differential and integro-differential equations.

5. **Mixed Problems Involving One or More Parameters.**—In view of the basic algebraic theory and the transcendental problems treated in §§ 1–4, it is clear that there must exist in the theory of integral equations expansion problems involving n parameters not only in the classic case when $n = 1$ but also in the general case when n is any positive integer. It was my intention to exhibit the basic formulæ on which such an expansion theory may be based; but, while this paper was in course of preparation, an abstract

* It should be observed that the negative values of n merely give us the same solutions over again. It is also worth noting that $u_n(x)/u_1(x)$ is a rational function of x so that from the expansion of $f(x)$ in terms of $u_n(x)$ we have that of $f(x)/u_1(x)$ in terms of a certain interesting set of rational functions.

† Here $\Delta f(x, s) = f(x+1, s) - f(x, s)$.

of a paper by Mrs. A. J. Pell appeared in the *Bulletin of the American Mathematical Society*, Ser. 2, Vol. 26 (1920), p. 149, indicating the character of this expansion problem for the case of two parameters. Consequently we shall leave out the formulæ for this case.

If we think of the several types of expansion problems—those for differential, difference, and integral equations—in intimate connection with the basic algebraic theory (developed in the memoir already referred to), it becomes apparent at once that the case of n parameters (for $n > 1$) is not confined to a set of n equations of the same sort. There is nothing to prevent one subset of the basic algebraic equations proceeding to differential equations as limiting forms, another to difference equations, another to integral equations, and so on. Thus we can see beforehand that we may formulate the expansion problem for a variety of mixed systems. In fact, we were led incidentally in § 2 to a special case of such mixed problems.

Let us now indicate more precisely the nature of these mixed problems by considering the equations

$$(1) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} (a_{ij} + \lambda b_{ij} + \mu c_{ij} + \nu d_{ij}) y_j, \quad i = 1, 2, \cdots, n,$$

$$(2) \quad \Delta s_i(x) = \sum_{j=1}^{m} (\varphi_{ij} + \lambda \rho_{ij} + \mu \sigma_{ij} + \nu \tau_{ij}) s_j(x), \quad i = 1, 2, \cdots, m,$$

$$(3) \qquad u(x) = \lambda \int_a^b K(x, \xi) u(\xi) d\xi + \mu \int_a^b L(x, \xi) u(\xi) d\xi + \nu \int_a^b M(x, \xi) u(x, \xi) d\xi,$$

associating with them their adjoint equations

$$(4) \qquad \frac{dz_i}{dx} = \sum_{j=1}^{n} (- a_{ji} - \lambda b_{ji} - \mu c_{ji} - \nu d_{ji}) z_j, \quad i = 1, 2, \cdots, n,$$

$$(5) \quad \Delta t_i(x) = \sum_{j=1}^{m} (- \varphi_{ji} - \lambda \rho_{ji} - \mu \sigma_{ji} - \nu \tau_{ji}) t_j(x + 1), \quad i = 1, 2, \cdots, m,$$

$$(6) \qquad v(x) = \lambda \int_a^b K(\xi, x) v(\xi) d\xi + \mu \int_a^b L(\xi, x) v(\xi) d\xi + \nu \int_a^b M(\xi, x) v(\xi) d\xi.$$

We suppose that the φ, ρ, σ, τ functions are analytic at infinity and vanish there to at least the second order.

From (1) and (4), from (2) and (5), by methods already frequently illustrated, we have

$$(7) \qquad [y_1 z_1 + y_2 z_2 + \cdots + y_n z_n]_{x=a'}^{x=b'} = 0,$$

$$(8) \qquad \sum_{i=1}^{m} [s_i(\infty) t_i(\infty) - s_i(\alpha) t_i(\alpha)] = 0.$$

We are now to set up homogeneous linear boundary conditions on y and on z so that (7) is implied by them; and on s and on t so that (8) is implied by them. We shall suppose that this is done in such a way that the sets of characteristic values for the two problems [(1), (2), (3) on the one hand

and (4), (5), (6) on the other] are the same and that the number of the sets is enumerably infinite. Then we shall denote the corresponding values and solutions by

$$\lambda^{(k)},\ \mu^{(k)},\ \nu^{(k)},\ y^{(k)},\ z^{(k)},\ s_i^{(k)},\ t_i^{(k)},\ u^{(k)},\ v^{(k)},\quad k = 1, 2, 3, \cdots.$$

In equations (1), (2), (3) let us replace the symbols for the parameters and the functions sought by the corresponding symbols with the superscript k; in equations (4), (5), (6) let us introduce similarly the distinct superscript l; and let us denote in order the six new equations so formed by $(\bar{1}), (\bar{2}), \cdots, (\bar{6})$. From $(\bar{1}), (\bar{4})$ and the boundary conditions implying (7) we have readily

$$(9)\quad \begin{aligned} &(\lambda^{(k)} - \lambda^{(l)})\int_a^b \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}y_i^{(k)}z_j^{(l)}dx + (\mu^{(k)}-\mu^{(l)})\int_a^b \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}y_i^{(k)}z_j^{(l)}dx \\ &\qquad + (\nu^{(k)} - \nu^{(l)})\int_a^b \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}y_i^{(k)}z_j^{(l)}dx = 0. \end{aligned}$$

Similarly, from $(\bar{2}), (\bar{5})$ and the boundary conditions implying (8), we have

$$(10)\quad \begin{aligned} &(\lambda^{(k)} - \lambda^{(l)})\sum_{w=0}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{m} \rho_{ji}(\alpha + w)s_i(\alpha + w)t_j(\alpha + 1 + w) + \cdots \\ &\qquad + (\nu^{(k)} - \nu^{(l)})\sum_{w=0}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{m} \tau_{ji}(\alpha + w)s_i(\alpha + w)t_j(\alpha + 1 + w) = 0. \end{aligned}$$

From $(\bar{3})$ and $(\bar{6})$ we get likewise

$$(11)\quad \begin{aligned} &(\lambda^{(k)} - \lambda^{(l)})\int_a^b\int_a^b K(\xi, x)u^{(k)}(x)v^{(l)}(\xi)dxd\xi + \cdots \\ &\qquad + (\nu^{(k)} - \nu^{(l)})\int_a^b\int_a^b M(\xi, x)u^{(k)}(x)v^{(l)}(\xi)dxd\xi = 0. \end{aligned}$$

Since the parametric differences $\lambda^{(k)} - \lambda^{(l)}, \cdots$, involved in the last three equations are not simultaneously zero when $k \neq l$ the determinant of the coefficients of these differences must have the value zero when $k \neq l$. The relations thus arising express the generalized biorthogonality property of the solutions pertaining to the problem in hand. Once these are obtained the formal determination of the coefficients in the formal expansions

$$(12)\quad f_{ij}(x_1, x_2, x_3) = \sum_{k=1}^{\infty} c_k y_i^{(k)}(x_1)s_j^{(k)}(x_2)u^{(k)}(x_3),\qquad i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m,$$

of a set of mn functions of three variables is easily effected by the method already employed several times in this paper.

The foregoing we have outlined briefly as an instance of a sort of thing which may be done in a great variety of ways. The formal aspects of the problem are easily developed whether we have a single equation of a given type, as in the problem here treated, or any number of equations of each type. Also, we may employ in the list of types integro-difference and

integro-differential equations. In view of the developments of the section following it will be seen that partial differential and partial difference equations are also admissible. Similarly one may allow equations involving functions of two variables and the operation of differentiation with respect to one and that of differencing with respect to the other; and so on. In this vast totality most types are without special interest; but a few actually arise in problems which come naturally to our attention.

Nor is this all. It is also possible to treat similarly equations in which functions are subjected to different operations with respect to the same variable. We illustrate this with one of the simplest cases. Let D_i denote a set of n operations, the first k of them being identical with D and the remaining ones with Δ, where k is some one of the numbers $0, 1, \cdots, n$, and where D is the symbol for differentiation and Δ is the symbol for the operation $\Delta f(x) = f(x+1) - f(x)$. Then form the system of equations

$$D_i u_i(x) = \sum_{j=1}^{n} (a_{ij} + \lambda b_{ij}) u_j(x), \quad i = 1, 2, \cdots, n, \tag{13}$$

and associate with them the adjoint system

$$D_i v_i(x) = \sum_{j=1}^{n} (- a_{ji} - \lambda b_{ji}) v_j(x + \epsilon_j), \quad i = 1, 2, \cdots, n, \tag{14}$$

where ϵ_i is zero or unity according as D_i is D or Δ. If we multiply (1) by $v_i(x + \epsilon_i)$ and (2) by $u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have a result which may be put in the form

$$\sum_{i=1}^{n} D_i \{ u_i(x) v_i(x) \} = 0. \tag{15}$$

Then, if appropriate conditions at infinity are satisfied, we have

$$\sum_{i=1}^{n} \{ u_i(\infty) v_i(\infty) - u_i(a) v_i(a) \} = 0.$$

It is clear now how one may complete the formulation of the expansion problem; the work proceeds in the closest analogy with special cases of it in §§ 2 and 4.

6. **Problems Arising from Partial Differential Equations and Integro-differential Equations.**—As typical of the general situation in regard to expansion problems arising from the theory of partial differential equations, let us consider that which is to be associated with the equation

$$L(u) + \lambda L_1(u) = 0 \tag{1}$$

and its adjoint

$$M(v) + \lambda M_1(v) = 0,$$

where

$$L(u) \equiv a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u,$$

$$L_1(u) \equiv \beta\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + ru,$$

$$M(v) \equiv \frac{\partial^2}{\partial x^2}(av) + 2\frac{\partial^2}{\partial x \partial y}(bv) + \frac{\partial^2}{\partial y^2}(cv) - \frac{\partial}{\partial x}(\alpha v) - \frac{\partial}{\partial y}(\beta v) + \gamma v,$$

$$M_1(v) \equiv -\frac{\partial}{\partial x}(pv) - \frac{\partial}{\partial y}(qv) + ru,$$

where the symbols $u, v, a, b, c, \alpha, \beta, \gamma, p, q, r$ denote functions of the independent variables x and y.

Let us multiply both members of equation (1) by v and of equation (2) by $-u$ and add the resulting equations member by member; thus we have

$$\begin{aligned}(3)\qquad &\frac{\partial}{\partial x}\left[a\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + b\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(\alpha + \lambda p - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)uv\right] \\ &\quad + \frac{\partial}{\partial y}\left[c\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + b\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial y}\right)\right. \\ &\qquad\qquad \left. + \left(\beta + \lambda q - \frac{\partial c}{\partial y} - \frac{\partial b}{\partial x}\right)uv\right] = 0.\end{aligned}$$

Let S be any closed region in the xy-plane whose boundary is cut twice and only twice by a line through it and parallel to either axis. Then if we integrate over S both members of equation (3) we have a relation which reduces readily to the form

$$\begin{aligned}(4)\qquad &\int_\rho^\sigma \left[a\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + b\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(\alpha + \lambda p - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)uv\right]_{x=r_y}^{x=s_y} dy \\ &\quad + \int_r^s \left[c\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + b\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)\right. \\ &\qquad\qquad \left. + \left(\beta + \lambda q - \frac{\partial c}{\partial y} - \frac{\partial b}{\partial x}\right)uv\right]_{y=\rho_x}^{y=\sigma_x} dx = 0,\end{aligned}$$

where r_y and s_y are limiting values of x in S for a given y and r and s are the extreme values of x and where $\rho_x, \sigma_x, \rho, \sigma$ have like meanings under interchange of rôles of x and y.

If in the first and second integrals in (4) we introduce the variable t in place of the given variable of integration by means of the relations

$$t = \frac{y - \rho}{\sigma - \rho}, \qquad t = \frac{x - r}{s - r},$$

respectively, each integral reduces to an integral with respect to t from

0 to 1, and hence (4) itself may be put in the form

$$\int_0^1 \mathrm{B}\,dt = 0 \tag{5}$$

where B is a sort of bilinear form in the functions u and v and their first derivatives, the arguments of the functions being specialized in the way implied by the transformations involved (so that each of them is a function of a single variable t).

We now require linear homogeneous boundary conditions on u and v separately, independent of λ, such that equation (5) is implied by the boundary conditions alone. We shall presently make clear the nature of these conditions by the explicit treatment of a particular case. For the present we shall suppose that these conditions have been set up in such way that the u-problem and the v-problem have the same characteristic values λ and that these are enumerably infinite in number. This done, we denote the characteristic values λ and the corresponding solutions by λ_i, u_i, v_i; $i = 1, 2, 3, \cdots$.

From (1) and (2) we now have the relations

$$L(u_i) + \lambda_i L_1(u_i) = 0,$$
$$M(v_j) + \lambda_i M_1(v_j) = (\lambda_i - \lambda_j) M_1(v_j).$$

If we multiply the first of these equations member by member by v_j and the second by $-u_i$, add the resulting equations member by member, and then integrate over S, the resulting first member has the value zero on account of the boundary conditions, and hence the second member also has the value zero. Omitting the non-zero factor $\lambda_i - \lambda_j$, this gives us the relation

$$\iint_S u_i M_1(v_j)\,dx\,dy = 0, \quad i \neq j. \tag{6}$$

Similarly, we have

$$\iint_S v_i L_1(u_j)\,dx\,dy = 0, \quad i \neq j. \tag{7}$$

These two (equivalent) sets of relations express the generalized biorthogonality conditions for the problem in consideration.

If we suppose that the integral $\iint u_i M_1(v_i)\,dx\,dy$ taken over S is different from zero for every i we have a ready means of determining formally the coefficients c_k in the formal expansion

$$f(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y) \tag{8}$$

of a function $f(x, y)$ of two variables in terms of the functions u_k. In fact,

we obtain readily the relations

$$c_k = \frac{\int_S\int f(x, y)M_1(v_k)dxdy}{\int_S\int u_k(x, y)M_1(v_k)dxdy}, \quad k = 1, 2, \cdots. \tag{9}$$

In order both to make the method of procedure more explicit and to illustrate the greater simplicity of the self-adjoint case of the foregoing problem let us consider an instance associated with a class of equations of great importance in theoretical physics, namely, equations of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda(g + h)u = 0, \tag{10}$$

where g is a function of x alone and h is a function of y alone. For the region S we take now the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$; and we use the boundary conditions

$$\begin{aligned} u(t, 1) - u(t, 0) &= 0, \qquad & u(1, t) - u(0, t) &= 0, \\ u_y(t, 1) - u_y(t, 0) &= 0, \qquad & u_x(1, t) - u_x(0, t) &= 0, \end{aligned} \tag{11}$$

where the subscript y or x denotes the partial derivative with respect to this variable. It is easy to verify that problem (10), (11) is self-adjoint; that is to say, if one writes the same problem in v one finds that these two problems, one in v and one in u, satisfy those conditions already imposed in the more general problems associated with (1) and (2).

Equation (6) now has the special form

$$\int_0^1\int_0^1(g + h)u_iv_jdxdy = 0, \quad i \neq j, \tag{12}$$

so that for the coefficients c_k in (8) we have the simpler values

$$c_k = \frac{\int_0^1\int_0^1(g + h)u_k(x, y)f(x, y)dxdy}{\int_0^1\int_0^1(g + h)\{u_k(x, y)\}^2dxdy}, \quad k = 1, 2, 3, \cdots. \tag{13}$$

Instead of considering all the solutions of (10) which satisfy conditions (11) we may fix our attention on any convenient class of them, as for instance those solutions which can be written in the form

$$u(x, y) = \rho(x)\sigma(y), \tag{14}$$

a class of particular solutions of great importance in theoretical and applied mechanics. Substituting $u(x, y)$ of the form (14) into (10) and (11) we find that a constant μ exists such that

$$\rho''(x) + \lambda g(x)\rho(x) + \mu\rho(x) = 0, \quad \rho(1) - \rho(0) = 0, \quad \rho'(1) - \rho'(0) = 0, \tag{15}$$

$$\sigma''(y) + \lambda h(y)\sigma(y) - \mu\sigma(y) = 0, \quad \sigma(1) - \sigma(0) = 0, \quad \sigma'(1) - \sigma'(0) = 0. \tag{16}$$

Each of the last two differential systems is self-adjoint. Together they form [if written in a single variable x] a system of two differential equations (with boundary conditions) in two parameters λ and μ. General problems of this type have already been treated in § 3. It was through the emergence of the multiple parameter problem in connection with partial differential equations that I was first led to seek the formulation of the general multiple parameter problem for algebraic systems and also for the transcendental cases treated in this memoir.

Let us derive the equations for the characteristic values of λ and μ in equations (15) and (16). Let $\rho_1(x, \lambda, \mu)$ and $\rho_2(x, \lambda, \mu)$ be a fundamental system of solutions of the differential equation in (15) and seek a solution $\rho(x, \lambda, \mu)$ so that the boundary conditions shall be satisfied. Constants c_1 and c_2 are to exist such that

$$\rho(x, \lambda, \mu) = c_1\rho_1(x, \lambda, \mu) + c_2\rho_2(x, \lambda, \mu);$$

hence we have through use of the boundary conditions in (15) the relations

$$\begin{aligned} c_1\{\rho_1(1, \lambda, \mu) - \rho_1(0, \lambda, \mu)\} + c_2\{\rho_2(1, \lambda, \mu) - \rho_2(0, \lambda, \mu)\} &= 0, \\ c_1\{\rho_1'(1, \lambda, \mu) - \rho_1'(0, \lambda, \mu)\} + c_2\{\rho_2'(1, \lambda, \mu) - \rho_2'(0, \lambda, \mu)\} &= 0. \end{aligned}$$

The condition of consistency of these equations is that

$$\begin{vmatrix} \rho_1(1, \lambda, \mu) - \rho_1(0, \lambda, \mu) & \rho_2(1, \lambda, \mu) - \rho_2(0, \lambda, \mu) \\ \rho_1'(1, \lambda, \mu) - \rho_1'(0, \lambda, \mu) & \rho_2'(1, \lambda, \mu) - \rho_2'(0, \lambda, \mu) \end{vmatrix} = 0, \tag{17}$$

a condition on λ and μ alone. Similarly, if $\sigma_1(y, \lambda, \mu)$ and $\sigma_2(y, \lambda, \mu)$ are a fundamental system of solutions of the differential equation in (16), we are led to the following second condition on λ and μ:

$$\begin{vmatrix} \sigma_1(1, \lambda, \mu) - \sigma_1(0, \lambda, \mu) & \sigma_2(1, \lambda, \mu) - \sigma_2(0, \lambda, \mu) \\ \sigma_1'(1, \lambda, \mu) - \sigma_1'(0, \lambda, \mu) & \sigma_2'(1, \lambda, \mu) - \sigma_2'(0, \lambda, \mu) \end{vmatrix} = 0. \tag{18}$$

The common solutions of equations (17) and (18) are the characteristic values for problem (15), (16). They are also the characteristic values for problem (10), (11) when the solutions u are restricted to be of the form given in (14).

Let us consider for a moment the special case in which $g = h = 1$. Then equations (15) and (16) have solutions (among others) of the form

$$\rho = e^{\alpha x}, \qquad \sigma = e^{\beta y},$$

where α and β are constants. The boundary conditions then require that we must have

$$e^{\alpha} - 1 = 0, \qquad \alpha(e^{\alpha} - 1) = 0, \qquad e^{\beta} - 1 = 0, \qquad \beta(e^{\beta} - 1) = 0.$$

Hence $\alpha = 2m\pi i$ and $\beta = 2n\pi i$ where m and n are integers, positive or

negative or zero. Then corresponding characteristic values λ and μ [perhaps not the complete set] are given by the relations $\lambda + \mu = -4m^2\pi^2$ and $\lambda - \mu = -4n^2\pi^2$, or

$$\lambda = -2\pi^2(m^2 + n^2), \qquad \mu = -2\pi^2(m^2 - n^2),$$

where m and n are integers. The corresponding solutions $u(x, y)$ may now be written

$$u_{mn}(x, y) = e^{2m\pi ix}e^{2n\pi iy}.$$

The expansion of $f(x, y)$ may be written

$$f(x, y) = \sum_{m, n=-\infty}^{+\infty} c_{mn}e^{2m\pi ix}e^{2n\pi iy}.$$

For a real-valued function $f(x, y)$ it is easy to transform this into an expansion in sines and cosines of integral multiples of x and y with real coefficients.

Let us consider the extension of our problem to adjoint systems of partial differential equations involving a single parameter. As a typical instance we take the system

$$\begin{aligned}(19)\quad \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} = \sum_{j=1}^{n}\left(a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + c_{ij}u_j\right)& \\ + \lambda\sum_{j=1}^{n}\left(\alpha_{ij}\frac{\partial u_j}{\partial x} + \beta_{ij}\frac{\partial u_j}{\partial y} + \gamma_{ij}u_j\right)&, \quad i = 1, 2, \cdots, n,\end{aligned}$$

and its adjoint system

$$\begin{aligned}(20)\quad \frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial y^2} = \sum_{j=1}^{n}\left[-a_{ji}\frac{\partial v_j}{\partial x} - b_{ji}\frac{\partial v_j}{\partial y} + \left(c_{ji} - \frac{\partial a_{ji}}{\partial x} - \frac{\partial b_{ji}}{\partial y}\right)v_j\right]& \\ + \lambda\sum_{j=1}^{n}\left[-\alpha_{ji}\frac{\partial v_j}{\partial x} - \beta_{ji}\frac{\partial v_j}{\partial y} + \left(\gamma_{ji} - \frac{\partial \alpha_{ji}}{\partial x} - \frac{\partial \beta_{ji}}{\partial y}\right)v_j\right]&, \quad i = 1, 2, \cdots, n.\end{aligned}$$

Let us multiply the first of these by v_i and the second by $-u_i$, add the resulting equations member by member, and sum as to i from 1 to n; thus we have

$$\begin{aligned}(21)\quad \sum_{i=1}^{n}\left[\frac{\partial}{\partial x}\left(v_i\frac{\partial u_i}{\partial x} - u_i\frac{\partial v_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(v_i\frac{\partial u_i}{\partial y} - u_i\frac{\partial v_i}{\partial y}\right)\right]& \\ - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial x}\left\{a_{ij} + \lambda\alpha_{ij}\right\}\cdot u_jv_i + \frac{\partial}{\partial y}\left\{b_{ij} + \lambda\beta_{ij}\right\}\cdot u_jv_i\right]& = 0.\end{aligned}$$

After the method employed in the first part of this section we may now obtain from (21) the condition which must be realized as a consequence of linear homogeneous boundary conditions on the u_i and the v_i separately. In (21) we integrate over a two-dimensional region S of the xy-plane, performing the integration with respect to x for those terms affected by the

operator $\partial/\partial x$ and with respect to y for those affected by the operator $\partial/\partial y$. In this way we have a relation analogous to (4) above. Just as in the preceding case we introduce a variable t so as to reduce the two integrals to a single integral of the form

$$\int_0^1 \mathrm{B}\,dt = 0 \tag{22}$$

where B is a sort of bilinear form. The boundary conditions must be set up so as to realize this relation; they may be set up so as to realize the more special relation B = 0.

Without attempting to characterize these boundary conditions in general [suggestions for forming them are afforded by the earlier part of this section and by the treatment of a like problem in § 1] we suppose that they have been chosen so that those on the u_i are linear and homogeneous in the u_i and their first derivatives and so that those on the v_i are linear and homogeneous in the v_i and their first derivatives and that this has been done in such way as to realize condition (22) as a consequence of the boundary conditions. We shall suppose also that the characteristic values λ_1, λ_2, λ_3, $\cdots$ of the u-problem are the same as those for the v-problem and that these values are infinite in number. The solutions corresponding to λ_k we denote by $u_i^{(k)}$, $v_i^{(k)}$.

In (19) let us write λ_k and $u_i^{(k)}$ for λ and u_i; in (20) let us write λ_l and $v_i^{(l)}$ for λ and v_i, where $k \neq l$. Multiplying the first of these resulting equations by $v_i^{(l)}$ and the second by $-u_i^{(k)}$, adding member by member, summing as to i from 1 to n, integrating as to x and y over the region S, simplifying by use of the relation to which (22) reduces when u_i is replaced by $u_i^{(k)}$ and v_i by $v_i^{(l)}$ [a relation which is valid in view of boundary conditions of the sort described], we have a relation which becomes the following when the non-zero factor $\lambda_k - \lambda_l$ is dropped.

$$\begin{aligned}\int\!\!\int_S \Bigg[\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg\{&-\alpha_{ji}\frac{\partial v_j^{(l)}}{\partial x}-\beta_{ji}\frac{\partial v_j^{(l)}}{\partial y}\\ &+\left(\gamma_{ji}-\frac{\partial\alpha_{ji}}{\partial x}-\frac{\partial\beta_{ji}}{\partial y}\right)v_j^{(l)}\Bigg\}u_i^{(k)}\Bigg]dxdy = 0, \quad k \neq l.\end{aligned} \tag{23}$$

Similarly we have the equivalent relation

$$\int\!\!\int_S \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x}+\beta_{ij}\frac{\partial u_j^{(k)}}{\partial y}+\gamma_{ij}u_j^{(k)}\right\}v_i^{(l)}\right]dxdy = 0, \quad k \neq l. \tag{24}$$

Now let $f_i(x, y)$, $i = 1, 2, \cdots, n$, be a system of n functions of the two variables x and y and let us seek simultaneous expansions of the form

$$f_i(x, y) = \sum_{\nu=1}^{\infty} c_\nu v_i^{(\nu)}(x, y), \quad i = 1, 2, \cdots, n, \tag{25}$$

where the c_ν are independent of i as well as of x and y. We seek to determine the c_ν on the supposition that the expression in the first member of (24) is different from zero whenever $k = l$. We multiply both members of (25) by

$$\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)},$$

sum both members as to j from 1 to n and as to i from 1 to n, and integrate in the resulting equation as to x and y over the region S. In view of (24) it is seen that c_k is the only coefficient left in the second member and that we have for c_k the value

$$(26)\quad c_k = \frac{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)}\right\} f_i(x, y)dxdy}{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)}\right\} v_i^{(k)}(x, y)dxdy},$$
$$k = 1, 2, \cdots.$$

In case $\alpha_{ij} \equiv 0 \equiv \beta_{ij}$ relations (24) and (26) reduce to the notably simpler forms

$$\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}u_j^{(k)}v_i^{(l)}dxdy = 0, \quad k \neq l,$$

$$c_k = \frac{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}u_j^{(k)}f_i dxdy}{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}u_j^{(k)}v_i^{(k)}dxdy}, \quad k = 1, 2, 3, \cdots.$$

Just as the problems in ordinary differential equations with one parameter can be generalized, as we have seen, to the case of any finite number of parameters so the problems treated in the foregoing part of this section can be extended to partial differential equations and systems in any finite number of parameters. Moreover, any finite number of independent variables may be employed in place of the two variables x and y of the foregoing treatment.

If we use the Volterra limiting process for passing from a finite number of differential equations to an integro-differential equation we are led from equations (19) and (20) above to certain adjoint partial integro-differential equations. For the sake of simplicity we shall confine our attention to the case in which the functions a_{ij}, b_{ij}, α_{ij}, β_{ij} are all identically zero. Then the integro-differential equations are the following:

$$(27)\quad \frac{\partial^2 u(x, y, s)}{\partial x^2} + \frac{\partial^2 u(x, y, s)}{\partial y^2} = \int_\alpha^\beta [c(x, y, s, t) + \lambda\gamma(x, y, s, t)]u(x, y, t)dt,$$

$$(28)\quad \frac{\partial^2 v(x, y, s)}{\partial x^2} + \frac{\partial^2 v(x, y, s)}{\partial y^2} = \int_\alpha^\beta [c(x, y, t, s) + \lambda\gamma(x, y, t, s)]v(x, y, t)dt.$$

These equations may be treated by processes closely analogous to those by which (19) and (20) were treated. We summarize the main results. Corresponding to (23) and (24) we now have

(29) $$\iint_S \left[\int_\alpha^\beta \int_\alpha^\beta \gamma(x, y, t, s) v^{(k)}(x, y, t) u^{(l)}(x, y, s) ds dt\right] dx dy = 0, \quad k \neq l.$$

Then if we seek expansions of the form

(30) $$f(x, y, s) = \sum_{\nu=1}^{\infty} c_\nu v^{(\nu)}(x, y, s)$$

we have for the coefficients c_k the values

(31) $$c_k = \frac{\iint_S \left[\int_\alpha^\beta \int_\alpha^\beta \gamma(x, y, s, t) u^{(k)}(x, y, t) f(x, y, s) ds dt\right] dx dy}{\iint_S \left[\int_\alpha^\beta \int_\alpha^\beta \gamma(x, y, s, t) u^{(k)}(x, y, t) v^{(k)}(x, y, s) ds dt\right] dx dy},$$
$$k = 1, 2, \cdots.$$

7. On a Problem Arising in the Theory of Vibrating Plates.—An important differential equation in the theory of vibrating plates is the following:

(1) $$\frac{\partial^4 \xi}{\partial x^4} + 2\frac{\partial^4 \xi}{\partial x^2 \partial y^2} + \frac{\partial^4 \xi}{\partial y^4} + \alpha \frac{\partial^2 \xi}{\partial t^2} = 0.$$

Here α is a constant and ξ is a function of x, y, t. In the theory of vibrating plates one desires to have a solution $\xi(x, y, t)$ of (1) which for $t = 0$ reduces to a given function $f(x, y)$ so that we have $\xi(x, y, 0) = f(x, y)$. Such solutions $\xi(x, y, t)$ have been found in a variety of ways in terms of particular solutions of the form

(2) $$\xi(x, y, t) = u(x, y)e^{int}$$

where n is a constant. If we substitute this value of ξ into (1) we have for u the equation

(3) $$L(u) + \lambda u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \lambda u = 0,$$

where $\lambda = -n^2\alpha$. We shall formulate the expansion problem as connected with equation (3) in the general form* suggested by the standard procedure which we follow under the guidance of the basic algebraic theory which directs our investigation.

The equation adjoint to the equation $L(u) + \lambda u = 0$ is obviously the equation $L(v) + \lambda(v) = 0$. We have the following readily verified basic identity:

* A number of special cases of this problem have been treated (with important results) in connection with the usual theory of vibrating plates.

Let us now write more explicitly

$$\text{(10)} \qquad B_1(x, y), \qquad B_2(x, y),$$

respectively, for the forms in the first and second brackets expressions in relation (4). Then for B we now have the value

$$\text{(11)} \qquad \begin{aligned} B = (d-c)\{B_1(b, c+(d-c)t) - B_1(a, c+(d-c)t)\} \\ + (b-a)\{B_2(a+(b-a)t, d) - B_2(a+(b-a)t, c)\}. \end{aligned}$$

Now let us think of $B_1(x, y)$ as a sort of bilinear form in the two sets of variables

$$\text{(12)} \qquad u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial x \partial y^2} \quad \text{and} \quad v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^3 v}{\partial x^3}, \frac{\partial^3 v}{\partial x \partial y^2}.$$

Let us form the matrix $||a_{ij}||$ of this bilinear form where the element a_{ij} of the ith column and jth row is the coefficient of the product of the ith function in the first set by the jth function in the second set. Then we have

$$||a_{ij}|| \equiv \begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & -1 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2-s & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ s-2 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

It is easy to show that this matrix is of rank 4; for the last row may be made to consist of elements zero by adding to it $(s-2)$ times the second last row, the fourth row may be made to consist of elements zero by adding to it $(s-2)$ times the third row, the last column may be made to consist of elements zero by subtracting from it s times the second last column, and the fourth column may be made to consist of elements zero by subtracting from it s times the third column, the matrix so obtained being the following (which obviously is of rank 4):

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

In $B_1(x, y)$ we have the six variables u and the six variables v listed in (12). In $B_2(x, y)$ we have similar six variables u and six variables v. The matrix of the bilinear form $B_2(x, y)$ is of rank 4, as one may readily verify by the method used for $B_1(x, y)$. Hence we may look upon B, defined as in (11), as a bilinear function of 24 variables u and 24 variables v, and this

bilinear form is of rank 16. Hence B may be put into the form

$$\mathrm{B} = \sum_{i=1}^{16} U_i(u) V_i(v), \tag{13}$$

where for each i $U_i(u)$ is a linear homogeneous function of the 24 variables u and $V_i(v)$ is a linear homogeneous function of the 24 variables v. The set $U_i(u)$, $i = 1, 2, \cdots, 16$, is linearly independent, and so also is the set $V_i(v)$, $i = 1, 2, \cdots, 16$. Then the boundary conditions which we take for the u-problem are

$$U_i(u) = 0, \quad i = 1, 2, \cdots, 8, \tag{14}$$

while those for the v-problem are

$$V_i(v) = 0, \quad i = 9, 10, \cdots, 16. \tag{15}$$

[For some parts of the theory one may take any number k of conditions $U_i(u) = 0$ not greater than 15, $i = 1, 2, \cdots, k$, and then use for the v-conditions $V_i(v) = 0, i = k + 1, \cdots, 16$. But the form of conditions given in (14) and (15) is needful for certain parts of the development.] If the conditions on v in (15) are the same as those on u in (14) the problem is self-adjoint and the maximum of elegance is secured.

UNIVERSITY OF ILLINOIS,
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RECIPROCITY IN A PROBLEM OF RELATIVE MAXIMA AND MINIMA.*

By James K. Whittemore.

Introduction.—Most students of the differential calculus have doubtless observed that the two following problems have the same solution: To determine the shape of a rectangle of given perimeter and *maximum* area; To determine the shape of a rectangle of given area and *minimum* perimeter. While in this and similar cases a quite elementary explanation may be given of the exchange of maximum and minimum corresponding to the exchange of area and perimeter it seems of interest to consider the analytical problem of maximum or minimum value of a function of two variables, subject to the condition that a second function of the two variables have a constant value, and to determine when the exchange of the rôles of the two functions results in the exchange of maximum and minimum. In this paper this question of reciprocity is discussed for the simplest problem, two functions of two variables. It is suggested that a similar discussion of the more general problem with more than two variables and with two or more than two functions might give results of considerable interest. The methods are analytical but are based largely on geometrical intuition. It is intended that the discussion should be complete, that is that all necessary and sufficient conditions should be given, in so far as this is possible with the use of the first and second partial derivatives of the two functions. In the first section the general case is considered, simple necessary and sufficient conditions for a relative extreme, maximum or minimum, obtained and stated in both analytical and geometrical form, and the reciprocity condition determined. In the second and third sections the more complicated exceptional cases are discussed. The fourth section is devoted to a study of the invariant properties of the conditions previously established. In the last section we give examples illustrating the theory developed in the preceding sections.

§1. **The General Case.**—Consider two functions φ and ψ of the two real variables, x and y; suppose that in the neighborhood of $P(x_0, y_0)$ both φ and ψ are single valued, real, and have continuous partial derivatives of the first and second orders. We write

$$\varphi(x, y) = u, \qquad \psi(x, y) = v,$$

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and denote the values of these functions at P by u_0 and v_0. We consider the two problems:

A. To determine when u_0 is an extreme of φ subject to the condition, $\psi = v_0$.

B. To determine when v_0 is an extreme of ψ subject to the condition, $\varphi = u_0$.

It is necessary to specify exactly what is meant by the statement, u_0 is, for example, a maximum of φ subject to the condition $\psi = v_0$. This statement shall in the following pages be understood to mean: (1) There exists a continuous set of real values x, y, of which set x_0, y_0 is an interior point, and of which all points satisfy the equation $\psi = v_0$; (2) For every point of this set, different from x_0, y_0, $u_0 > \varphi(x, y)$. Consider with this definition of relative maximum the example

$$\varphi = y^2, \qquad \psi = x^2 + y^2,$$
$$x_0 = y_0 = u_0 = v_0 = 0.$$

The condition, $x^2 + y^2 = 0$, gives $\varphi = -x^2$, and 0 is a maximum value of φ considered as a function of the independent real variable x, but according to the definition above is not a maximum relative to the condition, $x^2 + y^2 = 0$, for there is not a continuous set of *real* values, x, y, including 0, 0, satisfying the equation of condition. In some circumstances, as will appear in § 2, there are two or more sets of real values x, y satisfying the condition.* It may happen that u_0 is a maximum on both sets, supposing that there are two such sets, a maximum on one and not on the second, or not a maximum on either.

Evidently problems *A* and *B* are interchanged by exchanging φ and ψ. Problem *A* may be stated in geometrical form as follows: We regard x and y as ordinary rectangular coördinates in a plane which we find it convenient to call horizontal; the problem is that of an extreme of the ordinate u of the curve of intersection of the surface, S, $\varphi = u$, with the cylinder, $\psi = v_0$. Suppose first that P is not a singular point of the curve C in the xy plane, $\psi = v_0$; then one branch of C passes through P, and there is no essential restriction† in supposing that, for P, $\psi_y \neq 0$. Evidently a necessary condition for an extreme in *A* is that the tangent to the curve of intersection of S and cylinder at Q (x_0, y_0, u_0) be horizontal. For this curve

$$\frac{du}{dx} = \varphi_x + y'\varphi_y = 0, \qquad \frac{dv_0}{dx} = \psi_x + y'\psi_y = 0, \qquad y' = \frac{dy}{dx} = -\frac{\psi_x}{\psi_y}.$$

* Usually in the following pages we consider only such a set of points (x, y) as form a curve through P having at P a continuously turning tangent.

† See § 4.

It is necessary that, at P,

$$J = \varphi_x\psi_y - \psi_x\varphi_y = 0.$$

The "critical points," such as x_0, y_0, are determined in the usual elementary solution of A by solving simultaneously $\psi = v_0$ and $J = 0$.

For P we have

$$\frac{d^2u}{dx^2} = y''\varphi_y + y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx}, \qquad y'' = \frac{d^2y}{dx^2},$$

$$\frac{d^2v_0}{dx^2} = y''\psi_y + y'^2\psi_{yy} + 2y'\psi_{xy} + \psi_{xx} = 0.$$

If, at P, $\varphi_y = 0$ it follows from $J = 0$ that $\varphi_x = 0$. This case we consider in § 2, supposing now $\varphi_y \neq 0$. We have at P

$$\frac{d^2u}{dx^2} = \frac{\psi_x^2}{\psi_y^2}\varphi_{yy} - 2\frac{\psi_x}{\psi_y}\varphi_{xy} + \varphi_{xx} - \frac{\varphi_y}{\psi_y}\left(\frac{\psi_x^2}{\psi_y^2}\psi_{yy} - 2\frac{\psi_x}{\psi_y}\psi_{xy} + \psi_{xx}\right)$$
$$= \varphi_y\left[\frac{\varphi_x^2\varphi_{yy} - 2\varphi_x\varphi_y\varphi_{xy} + \varphi_y^2\varphi_{xx}}{\varphi_y^3} - \frac{\psi_x^2\psi_{yy} - 2\psi_x\psi_y\psi_{xy} + \psi_y^2\psi_{xx}}{\psi_y^3}\right],$$

since $J = 0$. We may write for P

$$\frac{d^2u}{dx^2} = \varphi_y(y_\psi'' - y_\varphi''),$$

where y_φ'' and y_ψ'' are the values at P of d^2y/dx^2 for the curves in the xy plane, $\varphi = u_0$ and $\psi = v_0$ respectively. For problem B we should evidently have at P the necessary condition $J = 0$, and

$$\frac{d^2v}{dx^2} = \psi_y(y_\varphi'' - y_\psi'').$$

We may state the following: With the hypothesis that each of the real curves in the xy plane, $\varphi = u_0$ and $\psi = v_0$, has an ordinary point at P, a condition necessary in both A and B is that these curves be tangent at P, $J = 0$; if this is the case a sufficient condition in both A and B is that the curves do not osculate at P; if both the necessary and sufficient conditions are satisfied A and B have like or unlike extremes as φ_y and ψ_y have different signs or the same sign at P. If the two curves osculate at P, so that the sufficient condition fails, problems A and B cannot be completely discussed by use of the first and second derivatives; we shall say in such a case the discussion fails.*

§ 2. **Exceptional Cases.**—We consider in this section the cases of problem A excluded in the preceding discussion. The conditions stated in each

* For example see § 5.

case are for the values x_0, y_0.

I. $\psi_x = \psi_y = 0$, φ_x and φ_y not both zero.

II. $\varphi_x = \varphi_y = 0$, ψ_x and ψ_y not both zero.

III. $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

To put the problem in geometrical form we consider x, y as before to be rectangular coördinates in a horizontal plane, and consider the intersection of the surface S with the cylinder, $\psi = v_0$, a cylinder with vertical elements erected on the curve, C, $\psi = v_0$, in the xy plane. This curve has in I and III a singular point at P. We assume that not all three of the second partial derivatives of ψ vanish at P, so that this point is a double point. This point is an isolated point with imaginary tangents, a cusp or osculating point or isolated point with one real tangent, or a point of intersection of two real branches of the curve with distinct real tangents, as $\Delta\psi = \psi_{xy}^2 - \psi_{xx}\psi_{yy}$ is negative, zero, or positive at P. The slopes of the tangents are the values of y' satisfying

$$y'^2\psi_{yy} + 2y'\psi_{xy} + \psi_{xx} = 0.$$

In II and III the tangent plane to S is horizontal at Q. The total curvature of the surface at this point is positive, zero, or negative, for all cases, as $\Delta\varphi$ is negative, zero, or positive, and the slopes of the horizontal projections of the asymptotic tangents at the point are in all cases given by the values of y' satisfying

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = 0.$$

In accordance with the limitations set for our discussion we suppose that not all of the second partial derivatives of φ vanish at Q, or, in geometrical terms, that Q is not a flat point of the surface S.

Case I.—$\psi_x = \psi_y = 0$, φ_x and φ_y not both zero. P is a double point of C; the tangent plane to the surface S at Q is not horizontal, but contains a single horizontal direction given by $y'\varphi_y + \varphi_x = 0$. We assume, as in the first section, $\varphi_y \neq 0$.

I*a*. $\Delta\psi < 0$. No real branch of C passes through P, and u_0 is not an extreme of φ.

I*b*. $\Delta\psi = 0$. Suppose the point P is a cusp or an osculating point of C. If P is a cusp and if the cuspidal tangent is not parallel to the horizontal tangent to S, at Q, that is if

$$D_\psi\varphi = \varphi_x^2\psi_{yy} - 2\varphi_x\varphi_y\psi_{xy} + \varphi_y^2\psi_{xx} \neq 0,$$

the ordinate u of a point moving on the curve of intersection of surface and cylinder will increase to u_0 then decrease, or decrease to u_0 then increase, since $du/dx \neq 0$ at P, and u_0 is an extreme of φ. The discussion fails in this case to distinguish between maximum and minimum. If P is an osculating

point of C and if a path with no cusp at P be followed a necessary condition for an extreme is $D_\psi \varphi = 0$. The discussion fails for further conditions, and fails also to distinguish between cusp, osculating point, and isolated point.

I*c*. $\Delta\psi > 0$. The point P is a double point of C with two real distinct tangents. In this case we consider only a path with a continuously turning tangent, that is a path projected into a branch of C. Evidently, for one branch of C, u_0 will not be an extreme of φ. In order that u_0 shall be an extreme for one branch of C it is necessary that $D_\psi \varphi = 0$. Further discussion of this case fails.

Case II.—$\varphi_x = \varphi_y = 0$, ψ_x and ψ_y not both zero. The point P is a simple point of C; the tangent plane to S at Q is horizontal. In all cases under II, since, at P, y_ψ'' is finite with the assumption $\psi_y \neq 0$,

$$\frac{d^2u}{dx^2} = \frac{D_\phi \psi}{\psi_y^2}.$$

II*a*. $\Delta\varphi < 0$. The total curvature of S is positive at Q, and u_0 is an absolute and consequently a relative extreme of φ, a maximum or minimum as $D_\phi \psi \neq 0$ is negative or positive. It is evident that φ_{xx} and φ_{yy} have the same sign as $D_\phi \psi$.

II*b*. $\Delta\varphi = 0$. The total curvature of S vanishes at Q, which is a parabolic point, since we have excluded the case of a flat point. There is at Q a single asymptotic tangent given by

$$y'^2 \varphi_{yy} + 2y' \varphi_{xy} + \varphi_{xx} = 0.$$

It is clear geometrically that u_0 is a relative extreme of φ if the tangent to C at P is not parallel to the asymptotic tangent, that is, if $D_\phi \psi \neq 0$. As in II*a* the nature of the extreme is given by the sign of $D_\phi \psi$, the same as that of φ_{xx} and φ_{yy} when neither of the latter vanishes. If $D_\phi \psi = 0$ the discussion fails.

II*c*. $\Delta\varphi > 0$. The total curvature of S is negative at Q, which is accordingly a hyperbolic point. There are at Q two distinct real asymptotic tangents given by the same equation as in II*b*. If the tangent to C at P is not parallel to either asymptotic tangent u_0 is an extreme of φ, whose nature is given as in II*a* and *b* by the sign of $D_\phi \psi$, but not by the sign of φ_{xx} or φ_{yy}. If $D_\phi \psi = 0$ the tangent at P is parallel to one of the asymptotic tangents and further discussion fails.

Case III.—$\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

The point P is a double point of C; the tangent plane to S at Q is horizontal.

III*a*. $\Delta\varphi < 0$. The total curvature of S is positive at Q.

1. $\Delta\psi < 0$. No real branch of C passes through P, and u_0 is not an extreme of φ.

2. $\Delta\psi = 0$, 3. $\Delta\psi > 0$. Since u_0 is an absolute extreme of φ it is a relative extreme for any path, a maximum or minimum as φ_{yy} is negative or positive, if P is not an isolated point of C in 2.

III*b*. $\Delta\varphi = 0$. The total curvature of S vanishes at Q, and there is a single asymptotic tangent.

1. $\Delta\psi < 0$. There is no extreme of φ at u_0.

2. $\Delta\psi = 0$. The curve C has a single tangent at the double point P. It is evident geometrically that u_0 is an extreme of φ if P is not an isolated point of C and if the tangent at P is not parallel to the asymptotic tangent at Q.

To express the last condition analytically consider the values at $P(x_0, y_0)$ of the two polynomials in y',

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = \varphi_{yy}(y' - \alpha_1)(y' - \alpha_2)$$
$$y'^2\psi_{yy} + 2y'\psi_{xy} + \psi_{xx} = \psi_{yy}(y' - \beta_1)(y' - \beta_2).$$

Their resultant R is

$$R = \varphi_{yy}^2\psi_{yy}^2(\alpha_1 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2) = H^2 - 4\Delta\varphi\Delta\psi,$$

where

$$H = \varphi_{xx}\psi_{yy} + \psi_{xx}\varphi_{yy} - 2\varphi_{xy}\psi_{xy}.$$

The condition that the tangents named be not parallel is $R \neq 0$, or since $\Delta\varphi = 0$, $H \neq 0$. As in III*a*, 2 and 3, the nature of the extreme is given by the sign of φ_{yy}, if $\varphi_{yy} \neq 0$; by the sign of φ_{xx} if $\varphi_{yy} = 0$; not both φ_{xx} and φ_{yy} vanish, and they have the same sign if neither vanishes. If $H = 0$ the two tangents are parallel and the discussion fails.

3. $\Delta\psi > 0$. The curve C has two real branches intersecting at P with distinct tangents. It is evident geometrically that u_0 is always an extreme of φ for one branch of C, and an extreme of the same kind for both branches if the tangent to neither branch is parallel to the asymptotic tangent to S at Q. Considered analytically, we have $R = H^2$, since $\Delta\varphi = 0$, and $\alpha_1 = \alpha_2$. If $H \neq 0$, we have for the two branches of C, since y_ψ'' is finite, d^2u/dx^2 equal to $\varphi_{yy}(\beta_1 - \alpha_1)^2$ and $\varphi_{yy}(\beta_2 - \alpha_1)^2$, $\varphi_{yy} \neq 0$; the two extremes are alike, maximum or minimum as φ_{yy} is negative or positive. If $H = 0$ the tangent to one branch of C is parallel to the asymptotic tangent at Q, and for this branch the discussion fails; for the other branch the nature of the extreme is given as before by the sign of φ_{yy}.

III*c*. $\Delta\varphi > 0$. The total curvature of S is negative at Q; there are two distinct asymptotic tangents.

1. $\Delta\psi < 0$. There is no extreme of φ at u_0.

2. $\Delta\psi = 0$. It is evident geometrically that sufficient conditions that u_0 be a relative extreme of φ are that P is not an isolated point of C and that the single tangent to C at P is not parallel to either asymptotic tangent at Q, that is $H \neq 0$. If $H = 0$ the discussion fails.

We determine the nature of the extreme when $H \neq 0$ as follows: In the neighborhood of x_0, y_0, we have

$$\varphi = u_0 + (\Delta y)^2\varphi_{yy} + 2\Delta x\Delta y\varphi_{xy} + (\Delta x)^2\varphi_{xx},$$

for values of the partial derivatives at a point near x_0, y_0. Since the derivatives are continuous by hypothesis u_0 is a maximum or minimum as

$$y'^2\varphi_{yy} + 2y'\varphi_{xy} + \varphi_{xx} = \varphi_{yy}(\beta_1 - \alpha_1)(\beta_1 - \alpha_2), \qquad y' = \beta_1 = \beta_2,$$

is negative or positive. Now it is easily proved that

$$H = \tfrac{1}{2}\varphi_{yy}\psi_{yy}\{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_1 - \beta_2)(\alpha_2 - \beta_1)\},$$

and, since in the case before us $\beta_1 = \beta_2$, we have

$$H = \varphi_{yy}\psi_{yy}(\alpha_1 - \beta_1)(\alpha_2 - \beta_1).$$

Then u_0 is a maximum or minimum as $\psi_{yy}H$ is negative or positive. The value of d^2u/dx^2 cannot be used in this discussion since in general, when $\Delta\psi = 0$, y_ψ'' becomes infinite.

3. $\Delta\psi > 0$. For both branches of C y_ψ'' is finite if $\psi_{yy} \neq 0$, and, if $\varphi_{yy} \neq 0$,

$$\frac{d^2u}{dx^2} = \varphi_{yy}(y' - \alpha_1)(y' - \alpha_2).$$

If $R < 0$, the values of this second derivative for the two branches of C,

$$\varphi_{yy}(\beta_1 - \alpha_1)(\beta_1 - \alpha_2), \qquad \varphi_{yy}(\beta_2 - \alpha_1)(\beta_2 - \alpha_2),$$

have opposite signs, and u_0 is an extreme of φ of opposite kinds for the two branches of C. If $R > 0$, the two values have the same sign and u_0 is an extreme of φ of the same kind for the two branches of C. If $R = 0$, one or both tangents at P are parallel to respectively one or both asymptotic tangents at Q. Along such a branch of C the discussion fails. If both tangents at P are parallel to asymptotic tangents at Q we must have

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

We consider the nature of the extremes of φ at u_0. Suppose first $R > 0$; there are two extremes of the same kind. If φ_{yy} and ψ_{yy} are both different from zero these extremes are maxima or minima as the following expression,

not zero, is negative or positive:

$$\varphi_{yy}\{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1) + (\alpha_1 - \beta_2)(\alpha_2 - \beta_2)\} = \frac{2}{\psi_{yy}^2} U_\phi,$$

$$U_\phi = \psi_{yy}H + 2\varphi_{yy}\Delta\psi.$$

The nature of the extremes is then given by the sign of U_ϕ. It will appear in § 4 that the restriction $\varphi_{yy}\psi_{yy} \neq 0$ is not essential. If $R = 0$, $\alpha_1 = \beta_1$, $\alpha_2 \neq \beta_2$, the criterion for maximum or minimum on the branch of C for which $y' = \beta_2$ is again the sign of U_ϕ, unless $\varphi_{yy} = \psi_{yy} = U_\phi = 0$, when the distinction will be given by the sign of

$$U_\phi' = \psi_{xx}H + 2\varphi_{xx}\Delta\psi \neq 0.$$

We shall show in § 4 that U_ϕ and U_ϕ' have the same sign when neither vanishes.

In three other cases, previously considered, the nature of the extreme or extremes is given by the sign of U_ϕ or of U_ϕ' when the former vanishes. These all come under III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$; they are b, $\Delta\varphi = 0$, 2, $\Delta\psi = 0$; b, 3, $\Delta\psi > 0$; c, $\Delta\varphi > 0$, 2, $\Delta\psi = 0$. In the discussion of b, 2, $H \neq 0$, it was shown that u_0 is a relative maximum or minimum as φ_{yy}, supposed different from zero, is negative or positive; in this case $U_\phi = \psi_{yy}H$ and this expression has the same sign as φ_{yy} if ψ_{yy} does not vanish, for

$$\varphi_{yy}\psi_{yy}H = \varphi_{yy}^2\psi_{yy}^2(\alpha_1 - \beta_1)^2 > 0.$$

In b, 3 the criterion was again the sign of φ_{yy}, which has the same sign as U_ϕ, when these are both different from zero, whether or not H vanishes; if either H or ψ_{yy} vanishes $U_\phi = \varphi_{yy}\Delta\psi$; if $\psi_{yy} \neq 0$, we have from the preceding paragraphs,

$$\varphi_{yy}\psi_{yy}^2\{(\alpha_1 - \beta_1)^2 + (\alpha_1 - \beta_2)^2\} = 2U_\phi.$$

In c, 2 it was proved that if $H \neq 0$ the extreme is maximum or minimum as $\psi_{yy}H$ is positive or negative; since $\Delta\psi = 0$, this is exactly U_ϕ. If $H = 0$, $U_\phi = U_\phi' = 0$, and the discussion fails. If $\Delta\varphi = \Delta\psi = R = 0$ we have the excluded case,

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

If $\Delta\varphi > 0$, $\Delta\psi = R = 0$ the discussion fails. In all other cases where

$$\Delta\varphi \geqq 0, \qquad \Delta\psi \geqq 0, \qquad R \geqq 0,$$

φ has at u_0 a maximum or a minimum as U_φ is negative or positive; if $U_\phi = 0$ the criterion is the sign of $U_\phi' \neq 0$. In certain cases φ has at u_0 two like extremes whose nature is given by the sign of U_ϕ or U_ϕ', namely

$\Delta\varphi \geqq 0$, $\Delta\psi > 0$, $R > 0$. In other cases φ has at u_0 one extreme whose nature is given by the sign of U_ϕ of U_ϕ', and may have a second extreme of either kind or no other extreme, namely $\Delta\varphi \geqq 0$, $\Delta\psi > 0$, $R = 0$.

§ 3. **Reciprocity in the Exceptional Cases.**—We now consider problems A and B together under the same assumed conditions with the purpose of determining, in so far as it is possible with the use of the first and second derivatives of φ and ψ, when extremes in the two problems are like or unlike.

It is clear that under assumed conditions for example in case I of § 2 the solution of B is given by interchanging φ and ψ in II.

In I, $\psi_x = \psi_y = 0$, φ_x and φ_y are not both zero. I*a*, $\Delta\psi < 0$. In A there is no extreme. For B the solution is given by II*a*, exchanging φ and ψ; there is always an extreme.

I*b*. $\Delta\psi = 0$. In A if P is not an isolated point of C a sufficient condition for an extreme is $D_\psi\varphi \neq 0$. For B the solution is that of II*b*; a sufficient condition is $D_\psi\varphi \neq 0$. The discussion fails for comparison of the nature of the extremes in A and B. The sufficient condition common to the two problems is, geometrically expressed, that the two curves, $\varphi = u_0$ and $\psi = v_0$, are not tangent at P.

I*c*. $\Delta\psi > 0$. In A it is necessary for an extreme that $D_\psi\varphi = 0$. For an extreme in B a sufficient condition, by II*c*, is $D_\psi\varphi \neq 0$.

The solutions of A and B with the assumed conditions of case II are given by exchanging φ and ψ in the preceding paragraphs.

In III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

III*a*. $\Delta\varphi < 0$. 1. $\Delta\psi < 0$. There is no extreme in A or B.

2. $\Delta\psi = 0$. There is always an extreme in A unless P is an isolated point of C. The solution of B is given by III*b*, 1; there is no extreme.

3. $\Delta\psi > 0$. There is an extreme for any path in A. In B, by III*c*, 1, there is no extreme.

III*b*. $\Delta\varphi = 0$. 2. $\Delta\psi = 0$. In both A and B, if P is not an isolated point of either curve, $\varphi = u_0$ or C, a sufficient condition for an extreme is $H \neq 0$. We prove that the extremes are like or unlike as H is positive or negative: Since $\Delta\varphi = \Delta\psi = 0$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$; since $R = H^2 \neq 0$, $\alpha_1 \neq \beta_1$ and $H = \varphi_{yy}\psi_{yy}(\alpha_1 - \beta_1)^2$. Then φ_{yy} and ψ_{yy} have like or unlike signs as H is positive or negative.

3. $\Delta\psi > 0$. In A there is always one extreme, and if $H \neq 0$ two like extremes. In B, if P is not an isolated point of the curve $\varphi = u_0$, a sufficient condition, by III*c*, 2, is $H \neq 0$. If $H \neq 0$ the extremes of A and B are like or unlike as φ_{yy} and $H\varphi_{yy}$ have like or unlike signs, that is as H is positive or negative.

III*b*. $\Delta\varphi > 0$. 3. $\Delta\psi > 0$. If $R < 0$ there are, as has been shown in the preceding section, two unlike extremes in problem A, and similarly two unlike extremes in B.

For $R = 0$, we have, with the notation of § 2, $\alpha_1 \neq \alpha_2$ since $\Delta\varphi > 0$, and $\beta_1 \neq \beta_2$ since $\Delta\psi > 0$, and $\alpha_1 = \beta_1$. If also $\alpha_2 = \beta_2$ then

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}},$$

and further discussion fails in both A and B. If these equations do not hold $\alpha_2 \neq \beta_2$ and there is certainly one extreme for both problems. For A and B respectively, if $\varphi_{yy}\psi_{yy} \neq 0$,

$$\frac{d^2u}{dx^2} = \varphi_{yy}(\beta_2 - \alpha_1)(\beta_2 - \alpha_2), \qquad \frac{d^2v}{dx^2} = \psi_{yy}(\alpha_2 - \beta_1)(\alpha_2 - \beta_2),$$

and since, for this case,

$$H = \tfrac{1}{2}\varphi_{yy}\psi_{yy}(\alpha_1 - \beta_2)(\alpha_2 - \beta_1),$$

the extremes known to exist in A and B are like or unlike as H is positive or negative.

For $R > 0$ there are two like extremes in A, similarly two like extremes in B. We shall show again that the extremes of A are like or unlike those of B as H is positive or negative. Supposing as before $\varphi_{yy}\psi_{yy} \neq 0$, the two values of d^2u/dx^2 for the two branches of C in A have the same sign and are, as given in § 2,

$$\varphi_{yy}(\beta_1 - \alpha_1)(\beta_1 - \alpha_2), \qquad \varphi_{yy}(\beta_2 - \alpha_1)(\beta_2 - \alpha_2).$$

The two values of d^2v/dx^2 in B are similarly

$$\psi_{yy}(\alpha_1 - \beta_1)(\alpha_1 - \beta_2), \qquad \psi_{yy}(\alpha_2 - \beta_1)(\alpha_2 - \beta_2)$$

and have the same sign. The signs of the two products,

$$\varphi_{yy}\psi_{yy}(\alpha_1 - \beta_1)(\alpha_2 - \beta_2), \qquad \varphi_{yy}\psi_{yy}(\alpha_1 - \beta_2)(\alpha_2 - \beta_1),$$

are alike and the same as the sign of H since the latter is one half their sum. Then the extremes of A and B are like or unlike as H is positive or negative.

We note that in all of the exceptional cases of § 2, where it is possible to determine the nature of the extremes, supposed alike when there are two, in both A and B by the use of the first and second derivatives of φ and ψ, the extremes in the two problems are the same or different as H is positive are negative. These all come under III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$, and include all cases where neither discriminant, $\Delta\varphi$ or $\Delta\psi$, nor the resultant R is negative, not all three vanish, and $H \neq 0$.

§ 4. **Invariant Properties.**—The various analytic conditions derived in the preceding sections must be independent of the geometrical representation of the problem and are accordingly invariant for any change of variables,

not singular at the point considered. It is of interest to consider directly the invariant properties of these conditions. It is also necessary to demonstrate the unessential character of assumptions as to non-vanishing of certain derivatives made in some of the discussions.

Suppose x and y are replaced in φ and ψ by two real functions of two new real variables, x' and y'. Concerning these functions we assume that both have for the point considered finite first and second partial derivatives; further, that at the same point the transformation is not singular, that is

$$\delta = x_{x'}y_{y'} - x_{y'}y_{x'} \neq 0.$$

For the new variables the various conditions will be obtained by replacing the derivatives of φ and ψ with respect to x and y in the conditions established by the corresponding derivatives of the same functions with respect to x' and y'. We have

$$\begin{aligned}
\varphi_{x'} &= \varphi_x x_{x'} + \varphi_y y_{x'}, \qquad \varphi_{y'} = \varphi_x x_{y'} + \varphi_y y_{y'}.\\
\varphi_{x'x'} &= \varphi_{xx} x_{x'}^{2} + 2\varphi_{xy} x_{x'} y_{x'} + \varphi_{yy} y_{x'}^{2} + \varphi_x x_{x'x'} + \varphi_y y_{x'x'},\\
\varphi_{x'y'} &= \varphi_{xx} x_{x'} x_{y'} + \varphi_{xy}(x_{x'}y_{y'} + x_{y'}y_{x'}) + \varphi_{yy} y_{x'} y_{y'} + \varphi_x x_{x'y'} + \varphi_y y_{x'y'},\\
\varphi_{y'y'} &= \varphi_{xx} x_{y'}^{2} + 2\varphi_{xy} x_{y'} y_{y'} + \varphi_{yy} y_{y'}^{2} + \varphi_x x_{y'y'} + \varphi_y y_{y'y'},
\end{aligned}$$

with similar equations for ψ.

Consider the quadratic form,

$$\begin{aligned}
Ap^2 + 2Bpq + Cq^2 &= A'p'^2 + 2B'p'q' + C'q'^2,\\
p = ap' + bq', &\qquad q = cp' + dq',\\
A' &= Aa^2 + 2Bac + Cc^2,\\
B' &= Aab + B(ad + bc) + Ccd.\\
C' &= Ab^2 + 2Bbd + Cd^2.
\end{aligned}$$

If we write

$$a = x_{x'}, \qquad b = x_{y'}, \qquad c = y_{x'}, \qquad d = y_{y'},$$

we may consider φ_x and φ_y as variables contragredient to p and q.* In the following discussion of the invariant properties of the conditions of the preceding sections it is to be remembered that the values of all derivatives of φ, ψ, x, and y considered are the values at $P\ (x_0, y_0)$. In the first place the classification of § 1 and in § 2, cases I, II, III is invariant, for, since $\delta \neq 0$, $\varphi_{x'} = \varphi_{y'} = 0$ if and only if $\varphi_x = \varphi_y = 0$. Next, if φ_x and φ_y are not both zero, and if ψ_x and ψ_y are not both zero it is clear that x' and y' may be chosen so that $\varphi_{y'}$ and $\psi_{y'}$ are both different from zero. The necessary condition of § 1, $J = 0$, is invariant for J is a contravariant* of weight one. The condition of the same section, sufficient for an extreme

* Bôcher, "Introduction to Higher Algebra," p. 109.

and determining the nature of the extreme, we consider later. When the sufficient condition is satisfied for both A and B the condition that the extremes are like or unlike is that φ_y and ψ_y have unlike or like signs. If ψ_y and $\psi_{y'}$ are both different from zero this condition is invariant if $J = 0$, for

$$\frac{\varphi_{y'}}{\psi_{y'}} = \frac{b\varphi_x + d\varphi_y}{b\psi_x + d\psi_y} = \frac{\varphi_y}{\psi_y}.$$

We may say that φ_y/ψ_y is an absolute conditional invariant, with condition $J = 0$.

In § 2 the further subdivision of the problem depends on the signs of $\Delta\varphi$ and $\Delta\psi$, $\Delta\varphi = \varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}$. It is evident from the values given above for the second derivatives of φ with respect to x' and y' that $\Delta\varphi$ is not in general invariant, but with the conditions, $\varphi_x = \varphi_y = 0$, the transformation is exactly the transformation of the quadratic form in p and q, and since the discriminant is an invariant of weight two in the algebraic theory,* we have conditionally $\Delta'\varphi = \delta^2\Delta\varphi$. The sign of $\Delta\varphi$ and the vanishing of $\Delta\varphi$, and similarly $\Delta\psi$, are therefore conditional invariants. If $\psi_x = \psi_y = 0$, the expression,

$$D_\psi\varphi = \varphi_x^2\psi_{yy} - 2\varphi_x\varphi_y\psi_{xy} + \varphi_y^2\psi_{xx},$$

is invariant in sign, for it is in the algebraic theory the form adjoint to $\psi_{xx}p^2 + 2\psi_{xy}pq + \psi_{yy}q^2$, and therefore a contravariant of weight two.†

We now consider the invariance of sign of $\varphi_y(y_\psi'' - y_\phi'')$, supposed not zero, in § 1. We may evidently write

$$\varphi_y(y_\psi'' - y_\phi'') = \frac{1}{\psi_y^2}\left(D_\phi\psi - \frac{\varphi_y}{\psi_y}D_\psi\psi\right),$$

where we suppose φ_y and ψ_y both different from zero, and $J = 0$. Under these conditions the sign of the expression is evidently unchanged by transformation if $\varphi_{y'}$ and $\psi_{y'}$ are also both different from zero, since φ_y/ψ_y is an absolute invariant, and since the terms of $D_\phi'\psi$ and $D_\psi'\psi$, which contain the second derivatives of x and y with respect to x' and y', cancel.

In case III we have the conditions, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$, so that the transformation is equivalent algebraically to that of two quadratic forms. We have supposed in that discussion that the following equations do not hold:

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}},$$

* Bôcher, l. c., p. 129.

† Bôcher, l. c., p. 159.

which include the special cases also excluded,

$$\varphi_{xx} = \varphi_{xy} = \varphi_{yy} = 0, \qquad \text{or} \qquad \psi_{xx} = \psi_{xy} = \psi_{yy} = 0.$$

That this condition is invariant when the four first derivatives vanish is obvious from the values given above for the second derivatives of φ with respect to x' and y'. From the same expressions it follows that under the assumed conditions it is possible so to choose x' and y' that neither φ_{yy} nor ψ_{yy} vanishes. In the same case we have stated that the nature of the extreme is in certain subcases dependent on the sign of φ_{yy} or ψ_{yy}. We have

$$\varphi_{y'y'} = \varphi_{xx}b^2 + 2\varphi_{xy}bd + \varphi_{yy}d^2,$$

and the signs of φ_{yy} and $\varphi_{y'y'}$ are the same when this form is definite, $\Delta\varphi < 0$, or, if neither vanishes, when the form is singular, $\Delta\varphi = 0$. Still considering the same case we have from the algebraic theory* the facts that H and R are invariants of weight two and four respectively, and are therefore unchanged in sign by transformation.

It remains only to consider the invariance of the sign of

$$U_\phi = \psi_{yy}H + 2\varphi_{yy}\Delta\psi,$$

the criterion for maximum or minimum in III, c, 3, $\Delta\varphi > 0$, $\Delta\psi > 0$; $R \geqq 0$. We recall the fact that both H and $\Delta\psi$ are invariants of weight two, and note that H does not vanish since $R = H^2 - 4\Delta\varphi\Delta\psi \geqq 0$. Now if φ_{yy} and ψ_{yy} are both different from zero it has been proved that the sign of U_ϕ is the same as that of

$$\varphi_{yy}\{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1) + (\alpha_1 - \beta_2)(\alpha_2 - \beta_2)\},$$

and $U_\phi \neq 0$. Evidently if one but not both of the derivatives, φ_{yy} and ψ_{yy}, vanishes $U_\phi \neq 0$. If also φ_{xx} and ψ_{xx} are both different from zero the sign of

$$U_\phi' = \psi_{xx}H + 2\varphi_{xx}\Delta\psi$$

is the same as that of

$$\varphi_{xx}\left\{\left(\frac{1}{\alpha_1} - \frac{1}{\beta_1}\right)\left(\frac{1}{\alpha_2} - \frac{1}{\beta_1}\right) + \left(\frac{1}{\alpha_1} - \frac{1}{\beta_2}\right)\left(\frac{1}{\alpha_2} - \frac{1}{\beta_2}\right)\right\}$$
$$= \varphi_{xx}\frac{\varphi_{yy}}{\varphi_{xx}}\left\{\frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1)}{\beta_1^2} + \frac{(\alpha_1 - \beta_2)(\alpha_2 - \beta_2)}{\beta_2^2}\right\},$$

and consequently in the cases before us, $R \geqq 0$, the same as that of U_ϕ. Any given non-singular transformation,

$$p = ap' + bq', \qquad q = cp' + dq',$$

* Bôcher, l. c., pp. 166, 236.

may be regarded as the transformation, $\theta = 1$, of the transformations θ,

$$p = a(\theta)p' + b(\theta)q', \qquad q = c(\theta)p' + d(\theta)q',$$

where the four functions of θ, a, b, c, d, are single-valued functions, continuous for $0 \leqq \theta \leqq 1$, and subject to the conditions:

$$a(0) = 1, \quad a(1) = a; \qquad b(0) = 0, \quad b(1) = b; \qquad c(0) = 0, \quad c(1) = c;$$
$$d(0) = 1, \quad d(1) = d.$$

Evidently $\theta = 0$ gives the identical transformation. If for the given transformation $ad - bc$ is positive the four functions of θ may be chosen so that in the interval named

$$a(\theta)d(\theta) - b(\theta)c(\theta)$$

does not vanish, and consequently no transformation θ is singular. For the same interval U_ϕ is a continuous function of θ. If $R > 0$ not both φ_{yy} and ψ_{yy} vanish for any θ, and U_ϕ has the same sign for $\theta = 0$ and $\theta = 1$. If for the given transformation $ad - bc$ is negative let us suppose, as we may without loss of generality, that originally no one of the four derivatives $\varphi_{xx}, \varphi_{yy}, \psi_{xx}, \psi_{yy}$ is zero. Let the transformation with $\delta = -1$,

$$x' = y, \qquad y' = x,$$

be first applied; U_ϕ is transformed without change of sign to U_ϕ', and may then be proved, as in the case of positive determinant, to be unchanged in sign by the given transformation.

The case $R = 0$, $\alpha_1 = \beta_1$, $\alpha_2 \neq \beta_2$, needs further consideration, for $U_\phi = 0$ if $\varphi_{yy} = \psi_{yy} = 0$. If however $U_\phi = 0$ necessarily $U_\phi' \neq 0$, for the possibility,

$$\varphi_{xx} = \varphi_{yy} = \psi_{xx} = \psi_{yy} = 0,$$

is excluded. It remains to show that U_ϕ and U_ϕ' have the same sign when neither is zero. Suppose, as before, that for $\theta = 0$ none of the four derivatives vanishes; then U_ϕ and U_ϕ' have the same sign. When θ increases from 0 to 1 U_ϕ and U_ϕ' have continually the same sign unless one, say U_ϕ, vanishes for $\theta = \theta_1$, when we must have $\varphi_{yy} = \psi_{yy} = 0$. It is now conceivable that, for further increase of θ, U_ϕ should change sign if either φ_{yy} or ψ_{yy} remains zero while the other is not zero. We prove that, with our assumptions, this cannot occur. Suppose, for $\theta = \theta_1$, $\varphi_{yy} = \psi_{yy} = 0$; that, for all values of θ such that $\theta - \theta_1$ is positive and less than some positive number ϵ, $\varphi_{yy} = 0$, $\psi_{yy} \neq 0$. From $R = 0$, we have

$$\varphi_{xx}^2\psi_{yy} - 4\varphi_{xx}\varphi_{xy}\psi_{xy} + 4\varphi_{xy}^2\psi_{xx} = 0,$$

for any value of θ between θ_1 and $\theta_1 + \epsilon$. This equation is satisfied by 1. $\varphi_{xx} = \psi_{xx} = 0$. This is inadmissible, since we cannot have, for θ_1, $\varphi_{xx} = \varphi_{yy} = \psi_{xx} = \psi_{yy} = 0$; 2. $\varphi_{xx} = \varphi_{xy} = 0$. This case is also excluded, for we cannot have, at θ_1, $\varphi_{xx} = \varphi_{xy} = \varphi_{yy} = 0$; 3. if $\varphi_{xx} \neq 0$,

$$\psi_{yy} = \frac{4\varphi_{xy}}{\varphi_{xx}^2}(\varphi_{xx}\psi_{xy} - \varphi_{xy}\psi_{xx}), \qquad \epsilon > \theta - \theta_1 > 0.$$

Since all the derivatives are continuous functions of θ and since, for θ_1, $\psi_{yy} = 0$ we should have in this case for θ_1, and consequently for all transformations, again the excluded case

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

Consequently our hypothesis is inadmissible, and U_ϕ is always of the same sign when it is different from zero. In the case III, b, 2, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0, \Delta\varphi = \Delta\psi = 0$; $H \neq 0$, we have $U_\phi = \psi_{yy}H$. It has been proved that ψ_{yy} cannot change sign though it may vanish, but not both ψ_{yy} and ψ_{xx} vanish. In b, 3, $\Delta\varphi = 0$, $\Delta\psi > 0$, we have, if $H = 0$, $U_\phi = \varphi_{yy}\Delta\psi$ and, since φ_{yy} may vanish but cannot change sign, the same is true of U_φ; not both φ_{yy} and φ_{xx} vanish. If $H \neq 0$ the proof given above for c, 3, $R > 0$, applies, and U_φ can neither vanish nor change sign. For c, 2, $\Delta\varphi > 0$, $\Delta\psi = 0$; $H \neq 0$, we have, as in b, 2, $U_\phi = \psi_{yy}H$. If $H = 0$ the discussion fails.

We remark that our discussion proves that U_ϕ and U_ψ have the same or different signs when neither is zero in the cases considered as H is positive or negative.

It is of interest to note that our discussion proves the existence of certain absolute conditional invariants or conditional differential parameters, for example

$$\frac{D_\phi\psi}{\Delta\varphi}, \qquad \frac{H}{\sqrt{\Delta\varphi\Delta\psi}}, \qquad \frac{\Delta\varphi}{\Delta\psi},$$

the first with the conditions, $\varphi_x = \varphi_y = 0$, the second and third with the conditions $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

§ 5. **Examples.**—We give in this section simple examples illustrating each case of the theory developed in the preceding sections, also examples showing, in each case where the discussion is said to fail, that there may exist a maximum or minimum or no extreme. We shall illustrate the reciprocity in problems A and B, and the invariance of the conditions in some of the examples by interchanging the variables x and y. All of the examples are so chosen that $x_0 = y_0 = u_0 = v_0 = 0$.

For the general case of § 1, φ_x and φ_y not both zero, ψ_x and ψ_y not both zero.

1. $$\varphi = y, \quad \psi = y - x^2.$$

We have $\varphi_y = \psi_y = 1$; $J = 0$; $y_\phi'' = 0$, $y_\psi'' = 2$. The curves, $y = 0$ and $y = x^2$, are tangent at $(0, 0)$ but do not osculate. In A we have a relative minimum, in B a relative maximum; φ_y and ψ_y have the same sign.

2. $$\varphi = y, \quad \psi = y - x^3, \quad y - x^4, \quad y + x^4.$$

For each ψ in this example the curves, $y = 0$ and $x = 0$, osculate at the origin; for the three choices φ has respectively no extreme, a minimum, a maximum.

For § 2, case I, φ_x and φ_y not both zero, $\psi_x = \psi_y = 0$.

a. $\Delta\psi < 0$.

3. $$\varphi = y, \quad \psi = x^2 + y^2.$$

Evidently no real branch of C, $\psi = 0$, passes through the origin.

b. $\Delta\psi = 0$. The point P $(0, 0)$ is a cusp, an osculating point, or an isolated point of C.

4. $$\varphi = y, \quad \psi = x^2 - y^3, x^2 + y^3, x^2 + y^4.$$

For the first two ψ, the origin is a cusp of C, and $D_\psi\varphi = 2$. For the first ψ there is a minimum, for the second a maximum of φ. For the third choice of ψ the origin is an isolated point of C and φ has no relative extreme.

5. $$\varphi = y, \quad \psi = x^3 - y^2, \quad (y - x^2)^2 - x^5, \quad (y + x^2)^2 - x^5.$$

For each choice of ψ the origin is a cusp of C, and $D_\psi\varphi = 0$. For the three choices φ has respectively no extreme, a minimum, a maximum.

6. $$\varphi = y, \quad \psi = y^2 - x^4 + x^5.$$

The origin is an osculating point of C, and $D_\psi\varphi = 0$. For a path on C with continuously turning tangent φ has at the origin no extreme, a minimum, or a maximum, depending on the path.

c. $\Delta\psi > 0$.

7. $$\varphi = y; \quad \psi = xy + x^4, \quad xy - x^3, \quad xy + x^3.$$

For each value of ψ the origin is a double point of C with two distinct tangents, and $D_\psi\varphi = 0$. If it be desired that C be irreducible y^2 may be added to each value of ψ without affecting the results. For the three choices of ψ the value zero is respectively no extreme, a minimum, a maximum of φ for that branch of C whose tangent is not parallel to the horizontal tangent to S at Q.

For § 2, case II, $\varphi_x = \varphi_y = 0$, ψ_x and ψ_y not both zero.

a. $\Delta\varphi < 0$.

8. $$\varphi = x^2 + y^2, \quad -x^2 - y^2, \qquad \psi = y.$$

The two choices of φ give $D_\phi\psi$ the values 2 and -2 respectively. For the first φ has a minimum, for the second a maximum.

b. $\Delta\varphi = 0$.

9. $$\varphi = x^2 - y^3, \quad x^2 + y^3, \qquad \psi = y.$$

Both choices of φ give $D_\phi\psi = 2$, and in both cases φ has a minimum. It is evident on comparing this example with example 4 that we have no basis for comparing the extremes of problems A and B with the hypotheses of I*b*. Similarly if we exchange φ and ψ as given in examples 5 and 6 we have for each $D_\phi\psi = 0$, from 5 no extreme and two minima, from 6 a maximum.

c. $\Delta\varphi > 0$.

10. $$\varphi = x^2 - y^2, \qquad \psi = x, y.$$

The two values of ψ give $D_\phi\psi$ equal to -2 and 2 respectively and give φ maximum and minimum values respectively. For problem B, corresponding to I*c* there is no extreme. If we interchange φ and ψ as given in example 7 we have D_φ and $\psi = 0$ and have a maximum in the first case, no extreme in the second and third. There is evidently no possibility of comparing the nature of the extremes in A and B in this case furnished by our discussion.

For § 2, case III, $\varphi_x = \varphi_y = \psi_x = \psi_y = 0$.

a. $\Delta\varphi < 0$. 1. $\Delta\psi < 0$.

11. $$\varphi = x^2 + y^2, \qquad \psi = 2x^2 + y^2.$$

There is no relative extreme in either A or B.

2. $\Delta\psi = 0$.

12. $$\varphi = x^2 + y^2.$$

Let ψ be given any of the values assigned in examples 4, 5, 6. For all choices φ has a minimum and $\varphi_{yy} = \varphi_{xx} = 2$.

3. $\Delta\psi > 0$.

13. $$\varphi = x^2 + y^2.$$

Let ψ have the values of 7. There is a minimum of φ for each branch of C.

b. $\Delta\varphi = 0$. 1. $\Delta\psi < 0$.

14. $$\varphi = y^2, \qquad \psi = x^2 + y^2.$$

Clearly φ has no relative extreme at the origin.

2. $\Delta\psi = 0$.

15. $$\varphi = y^2, \qquad \psi = x^2.$$

We find $\varphi_{yy} = 2$, $H = 4$, $U_\phi = 0$, $U_\phi' = 8$, $U_\psi = 8$, $U_\psi' = 0$.

Problems A and B have like extremes, both minima.

16. $\varphi = -y^2, \quad \psi = x^2.$

We have

$$\varphi_{yy} = -2, \quad H = -4, \quad U_\phi = 0, \quad U_\phi' = -8, \quad U_\psi = 8, \quad U_\psi' = 0.$$

Problems A and B have unlike extremes, a maximum in A, a minimum in B.

The case, $\Delta\varphi = \Delta\psi = R = H = 0$, is excluded, since, as previously stated, we have, when these all vanish,

$$\frac{\varphi_{xx}}{\psi_{xx}} = \frac{\varphi_{xy}}{\psi_{xy}} = \frac{\varphi_{yy}}{\psi_{yy}}.$$

The following example shows that in this case φ may have no extreme, a minimum or a maximum:

17. $\varphi = y^2, -y^2, \quad \psi = y^2 - x^3, \quad y^2 - x^4,$

3. $\Delta\psi > 0$.

18. $\varphi = y^2, \quad \psi = x^2 - y^2.$

We find

$$\begin{array}{llll} \varphi_{yy} = 2, & \psi_{yy} = -2, & \Delta\psi = 4, & H = 4, \\ U_\phi = 8, & U_\phi' = 8, & U_\psi = 8, & U_\psi' = 0. \end{array}$$

Problems A and B have like extremes, both minima.

19. $\varphi = y^2, \quad \psi = y^2 - x^2.$

We have

$$\begin{array}{llll} \varphi_{yy} = 2, & \psi_{yy} = 2, & \Delta\psi = 4, & H = -4, \\ U_\phi = 8, & U_\phi' = 8, & U_\psi = -8, & U_\psi' = 0. \end{array}$$

Problems A and B have unlike extremes, minima in A, a maximum in B.

20. $\varphi = y^2 + x^3, \quad y^2 + x^4, \quad y^2 - x^4, \qquad \psi = xy.$

We have for all choices of φ

$$\begin{array}{llll} \varphi_{yy} = 2, & \psi_{yy} = 0, & \Delta\psi = 1, & H = 0, \\ U_\phi = 4, & U_\phi' = 0, & U_\psi = 0, & U_\psi' = 0. \end{array}$$

Problem A has for the three choices of φ respectively a minimum and no extreme, two minima, a maximum and a minimum. In each case the nature of the known extreme, a minimum, is given by the positive sign of U_ϕ. The discussion fails for B.

c. $\Delta\varphi > 0$. 1. $\Delta\psi < 0$. There is no extreme in problem A.

2. $\Delta\psi = 0$.

21. $\varphi = x^2 - y^2, \quad \psi = y^2, \quad y^2 \pm x^3.$

We find $H = 4$, $U_\phi = 8$, $U_\phi' = 0$. There are a minimum in problem A, two minima in B. See example 18.

22. $\varphi = xy + x^3, \quad xy + x^4, \quad xy - x^4, \qquad \psi = y^2, \quad y^2 - x^8 + x^9.$

We have $H = U_\phi = U_\phi' = 0$. For the three choices of φ problem A has, for a path C with continuously turning tangent, respectively no extreme, a minimum, and a maximum. The discussion fails to give any information. See the reference to problem B in example 20.

3. $\Delta\psi > 0$.

23. $\varphi = x^2 - y^2, \qquad \psi = 2x^2 - 5xy + 2y^2.$

We find $\Delta\varphi = 4$, $\Delta\psi = 9$, $H = 0$, $R = -144$. In problem A the values of φ for the two branches of C are $3y^2$ and $-3x^2$, a minimum and a maximum respectively. In B the two values of ψ are $9y^2$ and $-x^2$, a minimum and a maximum.

24. $\varphi = x^2 - y^2, \qquad \psi = x^2 - 4y^2,$

We find

$$\begin{array}{llll} \Delta\varphi = 4, & \Delta\psi = 16, & H = -20, & R = 144, \\ U_\phi = 96, & U_\phi' = 24, & U_\psi = -24, & U_\psi' = -24. \end{array}$$

In problem A there are two like extremes, since $R > 0$, minima since $U_\phi > 0$, in B two like extremes, maxima since $U_\psi < 0$, unlike those of A since $H < 0$.

25. $\varphi = x^2 - y^2, \qquad \psi = 4y^2 - x^2.$

We have

$$\begin{array}{llll} \Delta\varphi = 4, & \Delta\psi = 16, & H = 20, & R = 144, \\ U_\phi = 96, & U_\phi' = 24, & U_\psi = 24, & U_\psi' = 24. \end{array}$$

There are two like extremes, both minima, in both A and B, of the same kind since $H > 0$.

26. $\varphi = x^2 - y^2 + x^3, \quad x^2 - y^2 \pm x^4, \qquad \psi = xy - y^2.$

We have

$$\begin{array}{llll} \Delta\varphi = 4, & \Delta\psi = 1, & H = -4, & R = 0, \\ & U_\phi = U_\phi' = 4, & U_\psi = U_\psi' = -8. & \end{array}$$

For that branch of C whose tangent is not parallel to an asymptotic tangent to S at Q, that is for $y = 0$, φ has a minimum value, since $U_\phi > 0$; for the other branch, $x - y = 0$, the three choices of φ give respectively no extreme, a minimum, and a maximum. In problem B the branch of the curve,

$\varphi = 0$, whose tangent, $x + y = 0$, is not parallel to an asymptotic tangent to the surface, $\psi = 0$, at the origin, gives ψ a maximum value, since $U_\psi < 0$, unlike the extreme in A since $H < 0$.

27. $$\varphi = x^2 - y^2, \qquad \psi = y^2 - xy.$$

We have

$$\Delta\varphi = 4, \qquad \Delta\psi = 1, \qquad H = 4, \qquad R = 0,$$
$$U_\phi = U_\phi' = 4, \qquad U_\psi = U_\psi' = 8.$$

In this case, since $H > 0$, the extremes, different in A and B in example 26, are the same, minima in both problems.

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